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Guest Commentary

Relativistic Thermodynamics for the Introductory Physics Course

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Abstract: *It is shown that the statistical development of entropy and the relativistic invariance of counted number of stable particles, offer an easy way to relativistic thermodynamics for the introductory physics course.*
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1. Introduction

Largely used introductory textbooks on special relativity do not treat the relativistic aspects of thermodynamics. The problem is treated in advanced textbooks and papers in a relatively complicated way using a four-vector approach, discouraging the teacher to use them when preparing a lecture on the subject. The situation is complicated by the fact that different transformation equations are proposed for two fundamental physical quantities: *temperature* and *energy transferred* as a result of a temperature difference (*heat*).

Fundamental equations in thermodynamics contain a number of N particles, which make up a thermodynamic system. The relativistic postulate: "all laws of physics are the same in all inertial reference frames" has the following important consequences:

- Universal constants (Planck's (h) and Boltzmann's (k_B)) and the counted number of stable particles have the same value in all inertial reference frames,
- If the quotient of two different physical quantities is a relativistic invariant, they should transform in accordance with the same transformation equation.

Let Φ and Ψ be two different physical quantities, measured in the $S(xOy)$ reference frame. Measuring them from an inertial reference frame $S'(x'O'y')$ they are Φ' and Ψ' respectively. If the two physical quantities satisfy the condition mentioned above

$$\frac{\Phi}{\Psi} = \frac{\Phi'}{\Psi'} \qquad , \tag{1}$$

then we should have

$$\Phi = f(v)\Phi' \qquad , \tag{2}$$

and

$$\Psi = f(v)\Psi' \qquad , \tag{3}$$

where $f(v)$ represents a function of the relative velocity v of the two reference frames. Relativistic kinematics, dynamics and electrodynamics offer examples of such functions known as transformation equations:

- A dimensionless combination of different physical quantities (mass, length, electric charge and Kelvin temperature, used in order to construct a system of units) is a relativistic invariant[1].

- If an equation is a combination of different physical quantities, one of them being a relativistic invariant, it enables us to construct an invariant combination of different physical quantities.
- If a physical law is described as a sum (or a difference) between different physical quantities then each of them should transform in accordance with the same transformation equation. Of course each term has the same physical dimensions. Knowing the transformation equation for one of them, we know how to transform the others ones. If one of the terms is a relativistic invariant then all the others have the same property.

If a transformation equation is derived for a particular case it should hold in general cases as well.

A system is considered[2], containing a large number N of particles, each of which can assume an energy from one of a discrete set of equally spaced energy levels. The energy of interaction is considered to be negligible. If the energy of a level j is $E_j = j\varepsilon$, being populated by n_j particles then we should have obviously

$$N = \sum_{j=0}^{\infty} n_j \tag{4}$$

$$E = \sum_{j=0}^{\infty} n_j E_j = \sum_{j=0}^{\infty} n_j j\varepsilon . \tag{5}$$

In relation (5) E represents the total energy of the system of noninteracting particles, called in macroscopic thermodynamics internal energy. In the same relation, j is the order number of a level and ε the distance between two consecutive levels measured in units of energy. A possible representation of such a system is shown in figure 1 below.

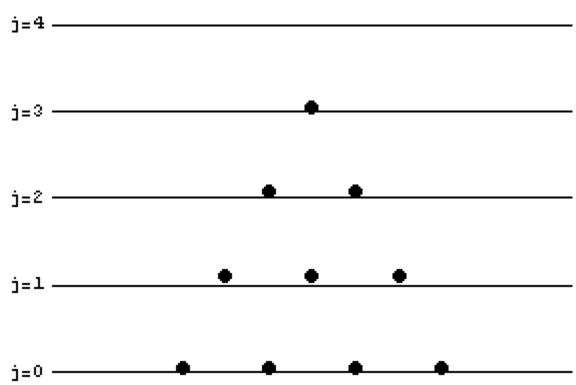

Figure 1. Distribution of the particles on the energy levels

Differentiating both sides of Equation (5) we obtain[3]

$$dE = \sum_{j=0}^{\infty} j\varepsilon dn_j + \sum_{j=0}^{\infty} jn_j d\varepsilon \tag{6}$$

In a statistical development of entropy it is considered that the energy that enters or leaves the system changes n_j, while ε remains constant. It is considered that this energy transfer takes place from a hotter object to a cooler one, which is often referred to by the *misnomer heat* [2,3]. Under such conditions Equation (6) becomes

$$dE = \sum_{j=0}^{\infty} j\varepsilon dn_j = \delta Q \tag{7}$$

with δQ representing the energy transferred under the conditions mentioned above. The statistical development of entropy leads to the following concepts useful in our approach to relativistic thermodynamics:

- **Number of microstates, *W***

$$W = \frac{N!}{n_0!n_1!n_2!n_3!...} \quad , \tag{8}$$

which is a relativistic invariant, being expressed as a function of counted numbers of stable particles,

- **entropy**

$$S = k_B \ln W \quad , \tag{9}$$

which is for the same reason a relativistic invariant as well,

- **change of entropy**

$$dS = k_B\, d \ln W = \frac{\delta Q}{T} \quad , \tag{10}$$

with T representing the Kelvin temperature of the system and δQ the transferred energy which changes n_j without to change ε. The transferred energy which changes ε without to change n_j is called work. As we see, $\frac{\delta Q}{T}$ is a relativistic invariant, δQ and T transforming in accordance with the same transformation equation *f(v).*

The fundamental physical quantities used in thermodynamics are *Kelvin temperature, heat, work, L* and internal *energy, U,* related by the first law of thermodynamics

$$\delta Q = dU + dL \quad , \tag{11}$$

with the sign conventions: heat introduced into the system is positive and the work done by the system is negative. Equation (11) suggests that δQ, dU, and dL should transform in accordance with the same transformation equation.

The invariance of $\frac{\delta Q}{T}$ requires that $\frac{dU}{T}$ and $\frac{dL}{T}$ are relativistic invariants as well. It is assumed that the student knows classical thermodynamics, relativistic kinematics, relativistic dynamics and relativistic electrodynamics as well.

2. Transformation equations for heat *Q* and Kelvin temperature, *T*

A. Exchange of heat at constant velocity

We consider a transfer of energy between two bodies 1 and 2 performed in a way called *heat* (radiation or existence of a temperature difference). *S'(x'O'y')* is the rest frame of the two bodies, both located on its *O'x'* axis (figure 2).

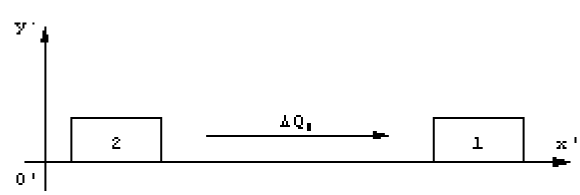

Figure 2. Energy exchange between bodies 1 and 2 at constant velocity in the *S'* reference frame where both of them are in a state of rest.

Let m_0 be the rest mass of body 1 and ΔQ_0 the energy (heat) it receives from body 2. During a given time interval its mass changes with

$$\Delta m_0 = \frac{\Delta Q_0}{c^2} \quad . \tag{12}$$

At a given instant of time the rest mass of body 1 is m_0^* ($m_0 < m_0^* < m_0 + \frac{\Delta Q_0}{c^2}$). We consider the same heat transfer process from the *S(xOy)* reference frame relative to which *S'* moves with constant velocity $v = \beta c$ in the positive direction of common *Ox(O'x')* axes. The momentum of the particle in the *S* frame at a given instant of time is g_x given by

$$g_x = \frac{m_0^* v}{\sqrt{1-\frac{v^2}{c^2}}} \quad , \tag{13}$$

resulting that the particle is acted upon by a force

$$F_x = \frac{dg_x}{dt} = \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dm_0^*}{dt} \quad , \tag{14}$$

where we have taken into account that the transfer of energy takes place at a constant velocity $(\frac{dv}{dt} = 0)$. The work done by that force is

$$dL = F_x v dt = \frac{v^2}{\sqrt{1-\frac{v^2}{c^2}}} dm_0^* . \tag{15}$$

Newton's second law applied in *S* leads to

$$\frac{d}{dt}(m^* v) = v\frac{dm^*}{dt} = v\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dm_0^*}{dt} . \tag{16}$$

In relation (16) m^* represents the instantaneous mass of particle 1 at a given instant of time. The change in the total energy of particle 1 is

$$dE = c^2 dm^* = c^2 \frac{dm_0^*}{\sqrt{1-\frac{v^2}{c^2}}} . \tag{17}$$

Combining Equations (12), (14) and (17) we obtain

$$dE = \delta L + \sqrt{1-\frac{v^2}{c^2}}\delta Q_0 . \tag{18}$$

First law of thermodynamics is in *S(xOy)*

$$dE = -\delta L + \delta Q \quad , \tag{19}$$

with δQ representing the transferred energy (heat) when measured from *S*. Comparing Equations (18) and (19) and taking into account the sign convention the result is

$$\delta Q = \sqrt{1-\frac{v^2}{c^2}}\delta Q_0 \quad , \tag{20}$$

a transformation equation for this energy transferred as heat and performed at a constant velocity of the heated body. Invariance of entropy requires that in this case temperature transforms as

$$T = \sqrt{1-\frac{v^2}{c^2}}T_0. \tag{21}$$

The results obtained above are in accordance with those proposed by Einstein[4] and Planck[5].

B. Heat transfer at constant momentum

Body 1 (spherical) at rest in S' and receives energy (heat) isotropically from the surrounding medium (*Fig.3*).

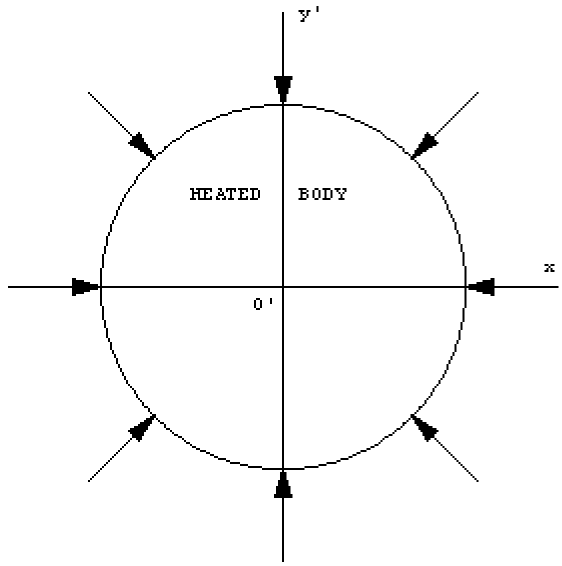

Figure 3. Energy transferred at constant momentum in the rest frame of the system of particles.

An increase in its energy

$$\Delta E^{'} = \Delta Q_0^{'} \tag{22}$$

takes place, its momentum, before and after the energy transfer being equal to zero. When detected from S the transferred energy is

$$\Delta Q = \frac{\Delta Q_0}{\sqrt{1-\frac{v^2}{c^2}}} \tag{23}$$

the invariance of entropy requires that in this case the temperature changes as

$$T = \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{24}$$

The results obtained above are in accordance with those obtained by Ott [6] and Arzélies [7].

3. The ideal gas and special relativity

The state of an ideal gas in thermodynamic equilibrium is described by the equation

$$P_0 V_0 = k_B N T_0 \qquad , \tag{25}$$

in the rest frame of its center of mass *S'*, P_0, V_0 and T_0 representing its pressure, volume and temperature respectively. Changes in its internal energy, heat transferred and work are related by Equation (19) owing that they are reversible. Reversibility requires that the changes in the state of the gas take place very slowly. This means that in S' *no change in the total momentum takes place, it being equal to zero at each moment of time.*

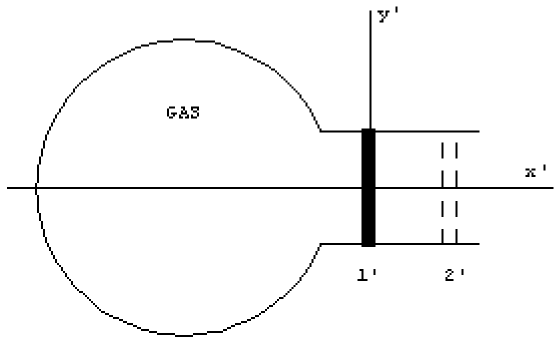

Figure 4. Scenario for deriving the equation of transformation for volumes.

A possible approach to the relativistic behavior of an ideal gas goes as follows: consider the vessel in which the gas is kept as it is shown in figure 4 as long as all the walls of the vessel are in a state of rest, the force $F_{0,x}$ which acts upon wall 1 normal to the axes *Ox(O'x')* as well as the surface of the wall are relativistic invariants and so is the pressure i.e.

$$P = P_0 \tag{26}$$

and is easy to shown that Equation (26) holds in the case of all the walls in accordance with the fact Equations (25) and (26) lead us to the following relativistic invariant

$$\frac{V_0}{T_0} = \frac{V}{T} \qquad , \tag{27}$$

owing that the volume of the gas transforms in accordance with the length contraction effect we have

$$V = V_0 \sqrt{1-\beta^2} \quad , \tag{28}$$

resulting that temperature should transform as

$$T = T_0 \frac{V}{V_0} = T_0 \sqrt{1-\beta^2} \quad . \tag{29}$$

Considering that work is done at constant temperature, work and heat are related by

$$L = Q = k_B N T \ln \frac{V_2}{V_1} \quad , \tag{30}$$

and the result is that work and heat transform as temperature does i.e.

$$L = \sqrt{1-\beta^2} L' \quad , \tag{31}$$

and

$$Q = \sqrt{1-\beta^2} Q' . \tag{32}$$

This is in accordance with the formulas obtained considering that heat transfer occurs at a constant velocity of the receiver. Equations (29) and (32) also save the invariance of entropy. In an adiabatic process we should have

$$P_0 = -\frac{\partial U_0}{\partial V_0}. \tag{33}$$

Invariance of pressure imposes the following transformation equation for volumes

$$dV = \sqrt{1-\beta^2}\, dV_0. \tag{34}$$

In a second approach to the same problem we involve the fact that all the thermodynamic processes we consider are reversible and if the total momentum of the gas is initially equal to zero in its rest frame, then it should remain equal to zero during the process (during the change of its macroscopic parameters) i.e. $G_0 = 0$. In accordance with the transformation equations for energy and momentum we have when detected from *S* frame

$$U = \gamma U_0 \tag{35}$$

$$G = \gamma \frac{v}{c^2} U_0 \quad . \tag{36}$$

U and U_0 representing the internal energies. The relativistic invariance of the first postulate requires that heat and work should transform as internal energy such as

$$L = \gamma L_0 \tag{37}$$

$$Q = \gamma Q_0 \quad , \tag{38}$$

where invariance of entropy requires that

$$T = \gamma T_0 \quad . \tag{39}$$

Equation (33) leads in that case to the transformation equation for elementary volumes

$$dV = \frac{dV_0}{\sqrt{1-\beta^2}}. \tag{40}$$

Of course the two approaches are not free of criticisms.

The weak point of the first approach is Equation (32), which implies for the internal energy an equation of transformation

$$dU = \gamma^{-1} dU_0 \quad , \tag{41}$$

which is not in accordance with the fact that the process is reversible. An apparent weak point of the second approach is Equation (40) which is conflicting with the length contraction effect.

But such a transformation equation is not strange to the physicist who knows that length contraction can lead sometimes to embarrassing paradoxes. Consider the scenario presented in figure 4. Let 1' and 2' be the initial and the final positions of a piston moving with constant velocity u' in the positive direction of the $O'x'$ axis. Its initial position is associated with the event $1'(x_1', y_1', t_1') = 1'(0,0,0)$ whereas its final position is associated with the event $2'(x_2', y_2', t_2') = 2'(u'dt',0,dt')$ resulting that displacement of the piston as measured from the *S* frame is

$$dx = \frac{u'dt' + vdt'}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{42}$$

The invariance of the surface of the piston leads to the following transformation equation for volumes

$$dV = \frac{dV_0}{\sqrt{1-\beta^2}} \quad , \tag{43}$$

owing that the involved time intervals are very small. A length measurement procedure proposed by Streltsov[8], involving clocks and light signals leads to Equation (43) as well.

Taking into account the fact that the first approach is not in accordance with the concept of reversibility and involves formulas which hold in the case in the case of energy transfer at constant velocity and not at constant momentum we incline to the view that the second approach is correct.

4. Conclusions

Contradictory formulas encountered in the field of relativistic thermodynamics are discussed showing the conditions under which they hold. The conclusion is that the formulas proposed by Planck[4] and Einstein[5] hold in the case when a transfer of energy takes place at constant velocity whereas those proposed by Ott[6] and Arzélies[7] hold in the case of a transfer of energy at constant total momentum. It was shown how they work in the case of an ideal gas underlining the advantages of the second approach.

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