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Comparative Study of Quaternions and Mixed Numbers

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Abstract: The general quaternion is X = a + bi + cj + dk where a, b, c, and d are real and the mixed number which is the sum of a scalar and a vector (i.e. $\alpha = x + \mathbf{A}$ where x is a scalar quantity and **A** is a vector quantity). In this paper we compare the mixed number with quaternion.

Keywords: quaternions, mixed numbers.

Introduction

The algebra of quaternions discovered [1] by the Irish mathematician Sir William Rowan Hamilton. Quaternions are "hypercomplex" [2] numbers with "imaginary" units i, j, k which satisfy the relations
$i^2 = j^2 = k^2 = -1$,
ij = k, jk = i, ki = j,
ij + ji = 0, ik + ki = 0, jk + kj = 0 .
The general quaternion is X = a + bi + cj + dk where a,b,c,d are real.

Mixed number [3] α as a sum of a scalar x and a vector **A**: $\alpha = x + \mathbf{A}$

Quaternion Algebra

Definition: For $X = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, $Y = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k$ the addition is defined [4] as
$X+Y = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)i + (\alpha_2 + \beta_2)j + (\alpha_3 + \beta_3)k$.

Definition: For $X = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, $Y = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k$ the multiplication is defined [4] as
$X.Y = (\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k).(\beta_0 + \beta_1 i + \beta_2 j + \beta_3 k) =$
$(\alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3) + (\alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2)i +$
$(\alpha_0\beta_2 + \alpha_2\beta_0 + \alpha_3\beta_1 - \alpha_1\beta_3)j + (\alpha_0\beta_3 + \alpha_3\beta_0 + \alpha_1\beta_2 - \alpha_2\beta_1)k$.
[Using the relations: $i^2 = j^2 = k^2 = -1$, ij = k, jk = i, ki = j, ij + ji = 0, ik + ki = 0, jk + kj = 0 .]

Definition: For $X = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ in Q the adjoint of X, denoted by X *, is defined [4] by X * =
$\alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$.

Lemma 1. The adjoint in Q satisfies [4]
(1) X * * = X

(2) $(\delta X + \gamma Y)^* = \delta X^* + \gamma Y^*$
(3) $(XY)^* = Y^* X^*$
for all X, Y in Q and all real δ and γ .

Definition: If $X \in Q$ then the norm of X, denoted by N(X), is defined [3] by $N(X) = XX^*$.

Lemma 2. For all $X, Y \in Q$, $N(XY) = N(X)N(Y)$.

Definition: Real Quaternions consist of all matrices of the form [5]

$$a1 + bi + cj + dk = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} .$$

Definition: The first known noncommutative division algebra was Hamilton’s quaternions.[6]

Definition: The product of real quaternoins are associative, i.e. $(xy)z = x(yz)$.

Mixed Number Algebra

Definition: For $\alpha = x + \mathbf{A}$ and $\beta = y + \mathbf{B}$ the addition is defined as
$\alpha + \beta = (x + y) + (\mathbf{A} + \mathbf{B})$.

Definition: For $\alpha = x + \mathbf{A}$ and $\beta = y + \mathbf{B}$ the product is defined as
$\alpha\beta = (x + \mathbf{A})(y + \mathbf{B}) = xy + \mathbf{A}.\mathbf{B} + x\mathbf{B} + y\mathbf{A} + i\mathbf{A}x\mathbf{B}$.

Definition: The adjiont of α, denoted by α^*, is defined by $\alpha^* = x - \mathbf{A}$.

Lemma 1. The adjoint in α satisfies
(i) $\alpha^{**} = \alpha$
(ii) $(\alpha + \beta)^* = \alpha^* + \beta^*$
(iii) $(\alpha\beta)^* = \beta^* \alpha^*$.

Proof. (i)
For $\alpha = x + \mathbf{A}$, $\alpha^* = (x + \mathbf{A})^* = x - \mathbf{A}$ $\therefore \alpha^{**} = (x - \mathbf{A})^* = x + \mathbf{A} = \alpha$

Proof. (ii)
For $\alpha = x + \mathbf{A}$ and $\beta = y + \mathbf{B}$, $\alpha + \beta = (x + y) + (\mathbf{A} + \mathbf{B})$
$\therefore (\alpha + \beta)^* = [(x + y) + (\mathbf{A} + \mathbf{B})]^* = (x + y) - (\mathbf{A} + \mathbf{B}) = (x - \mathbf{A}) + (y - \mathbf{B}) = \alpha^* + \beta^*$

Proof. (iii)
For $\alpha = x + \mathbf{A}$ and $\beta = y + \mathbf{B}$
$\alpha\beta = (x + \mathbf{A})(y + \mathbf{B}) = xy + \mathbf{A}.\mathbf{B} + x\mathbf{B} + y\mathbf{A} + i\mathbf{A}x\mathbf{B}$
$(\alpha\beta)^* = [(x + \mathbf{A})(y + \mathbf{B})]^* = [(xy + \mathbf{A}.\mathbf{B}) + (x\mathbf{B} + y\mathbf{A} + i\mathbf{A}x\mathbf{B})]^*$
$= (xy + \mathbf{A}.\mathbf{B}) - (x\mathbf{B} + y\mathbf{A} + i\mathbf{A}x\mathbf{B})$ .... (1)

Now $\beta^* \alpha^* = (y - \mathbf{B})(x - \mathbf{A}) = (yx + \mathbf{B}.\mathbf{A} - x\mathbf{B} - y\mathbf{A} + i\mathbf{B}\times\mathbf{A})$
$= [(xy + \mathbf{A}.\mathbf{B}) - (x\mathbf{B} + y\mathbf{A} + i\mathbf{A}\times\mathbf{B})]$ .... (2)

From equation (1) and (2) we can write $(\alpha\beta)^* = \beta^* \alpha^*$.

Definition: The norm of a Mixed number α, denoted by $N(\alpha)$, is defined by
$N(\alpha) = \alpha\alpha^* = (x + \mathbf{A})(x - \mathbf{A}) = x^2 - A^2 - x\mathbf{A} + x\mathbf{A} - i\mathbf{A}\times\mathbf{A} = x^2 - A^2$.

Lemma 1. For all α and β $N(\alpha\beta) = N(\alpha)N(\beta)$.

Proof.
For $\alpha = x + \mathbf{A}$ and $\beta = y + \mathbf{B}$
$\alpha\beta = (x + \mathbf{A})(y + \mathbf{B}) = xy + \mathbf{A}.\mathbf{B} + x\mathbf{B} + y\mathbf{A} + i\mathbf{A}x\mathbf{B}$

$\therefore N(\alpha\beta) = (xy + \mathbf{A}.\mathbf{B})^2 - (x\mathbf{B} + y\mathbf{A} + i\mathbf{A}x\mathbf{B})^2$
$= x^2y^2 + 2xy\mathbf{A}.\mathbf{B} + (\mathbf{A}.\mathbf{B})^2 - (x\mathbf{B} + y\mathbf{A})^2 - 2(x\mathbf{B} + y\mathbf{A}).i(\mathbf{A}x\mathbf{B}) - i^2(\mathbf{A}x\mathbf{B})^2$

$x^2y^2 + 2xy\mathbf{A}.\mathbf{B} + (\mathbf{A}.\mathbf{B})^2 - (x\mathbf{B} + y\mathbf{A})^2 + (\mathbf{A}x\mathbf{B})^2$
$= x^2y^2 + 2xy\mathbf{A}.\mathbf{B} + (\mathbf{A}.\mathbf{B})^2 - x^2B^2 - 2xy\mathbf{A}.\mathbf{B} - y^2A^2 + (\mathbf{A}x\mathbf{B})^2$
$= x^2y^2 + 2xy\mathbf{A}.\mathbf{B} + (\mathbf{A}.\mathbf{B})^2 - x^2B^2 - 2xy\mathbf{A}.\mathbf{B} - y^2A^2 - (\mathbf{A}x\mathbf{B})^2$
$= x^2y^2 + (\mathbf{A}.\mathbf{B})^2 - x^2B^2 - y^2A^2 - (\mathbf{A}x\mathbf{B})^2$
$= x^2y^2 - x^2B^2 - y^2A^2 + (\mathbf{A}.\mathbf{B})^2 - (\mathbf{A}x\mathbf{B})^2$
$= x^2y^2 - x^2B^2 - y^2A^2 + A^2B^2 = [(\mathbf{A}.\mathbf{B})^2 - (\mathbf{A}x\mathbf{B})^2 = A^2B^2$]
$=(x^2 - A^2)(y^2 - B^2) = N(\alpha)N(\beta)$

$\therefore N(\alpha\beta) = N(\alpha)N(\beta)$

Definition: The product of Mixed numbers are associative, i.e. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

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