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## The Relativistic Velocity Addition Law through Special Relativity

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**Abstract:** It is shown that if we can define a physical quantity with proper character in a given inertial reference frame (kinematic, dynamic, electromagnetic in its nature) which transforms when detected from a reference frame relative to which it moves with velocity  $u_x$  as  $F = \frac{F^0}{\sqrt{1 - \frac{u_x^2}{c^2}}}$ , then we can derive for it transformation equations following one and

the same procedure which involves the addition law of relativistic velocities which can be derived without using the Lorentz transformations. The transformation equation derived that way, generates the physical quantities  $u_x F$  and  $u'_x F'$ , for which physicists invent names reflecting their physical meaning.

**Keywords:** special relativity, relativistic velocity addition law, kinematics.

### 1. Introduction

Each chapter of physics operates with fundamental physical quantities. Kinematics is involved with length and time, dynamics add to them, by tradition, the concept of mass, whereas electromagnetism introduces the concept of electric charge. Combining fundamental physical quantities we obtain new ones, useful in describing physical effects.

All chapters of physics have in common the relativistic postulate, due to Galileo and Einstein, according to which all the true laws of physics are the same for all inertial observers, all of them measuring the value  $c$  for the velocity of light. When measured from different reference frames some physical quantities have the same magnitude (relativistic invariants). If a physical quantity which characterizes an object is measured by an observer relative to whom the object is in a state of rest, we say that the observer measures its proper value.

An equation which relates physical quantities measured from different reference frames, represents a transformation equation. A transformation equation enables us to derive formulas which account for different relativistic effects. It is considered that the use of the transformation equations reduces the transparency<sup>1,2</sup>.

Relativistic equations describing time dilation<sup>1,3,4,5</sup>, length contraction, and addition of relativistic velocities<sup>1,6,7</sup> can be derived without using transformation equations, all having as a starting point the time dilation formula. This method works when in one of the involved reference frames a proper physical quantity is being measured.

The purpose of our paper is to show that the transformation equations for kinematic, dynamic, and electromagnetic physical quantities can be derived using the same strategy, involving the addition law of relativistic velocities as a single relativistic ingredient, which is a direct consequence of the time dilation effect<sup>7</sup>.

Three inertial reference frames are involved  $S(XOY)$ ,  $S'(X'O'Y')$ , and  $S^0(X^0O^0Y^0)$ .  $S^0$  moves with velocity  $u_x$  relative to  $S$ , with velocity  $u'_x$  relative to  $S'$ ,  $S'$  moving at its turn with velocity  $v$  relative to  $S$ . The axes of the reference frames are parallel to each other, the  $OX$ ,  $O'X'$ , and  $O^0X^0$  axes are overlapped and all the velocities show in their positive directions. The three reference

frames have the same space-time origins. In order to keep the problem as simple as possible, we will consider only events which take place on the overlapped axes. The velocities mentioned above are related by

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} \quad (1)$$

which solved for  $u_x'$  leads to

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \quad (2)$$

Reference frame  $S^0(X^0O^0Y^0)$  is a special one. In that frame nothing and nobody moves and only events which take place at its origin  $O^0$  are taken into account. The result is that there, only physical quantities which characterize a stationary body can be detected, like the reading of clock  $C_0^0$  located at  $O^0(t^0)$ , the mass  $m^0$  of a particle located at  $O^0$ , the force  $F^0$  acting on the particle mentioned above and the electric field created by a charged capacitor. Physical quantities different from zero space coordinates, momentum, mechanical power, and magnetic field are meaningless in that frame. All the physical quantities measured in  $S^0$  are by definition proper physical quantities: proper time reading (date)  $t^0$ , proper mass  $m^0$ , proper force  $F^0$ , and proper electric field  $E^0$ .

## 2. Relativistic Kinematics

In each of the mentioned reference frames, the clocks located at the different points of the overlapped axes are synchronized following a procedure proposed by Einstein. The proposed scenario makes that at a given point of the overlapped axes, clock  $C_0^0$  located at  $O^0$  meets clock  $C(x, y = 0)$  of the  $S$  frame and a clock  $C'(x', y' = 0)$  of the  $S'$  frame. Let  $t^0$ ,  $t$  and  $t'$  be the readings of the three clocks respectively. In accordance with the time dilation formula the three readings are related by

$$t = \frac{t^0}{\sqrt{1 - \frac{u_x^2}{c^2}}} \quad (3)$$

and by

$$t' = \frac{t^0}{\sqrt{1 - \frac{u_x'^2}{c^2}}} \quad (4)$$

Eliminating  $t^0$  we obtain that  $t$  and  $t'$  are related by

$$t = t' \frac{\sqrt{1 - \frac{u_x'^2}{c^2}}}{\sqrt{1 - \frac{u_x^2}{c^2}}} = t' \frac{1 + \frac{u_x' v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t' + \frac{v(u_x' t')}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (5)$$

and by

$$t' = t \frac{\sqrt{1 - \frac{u_x^2}{c^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} = t \frac{1 - \frac{u_x v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t - \frac{v(u_x t)}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} . \quad (6)$$

We detect in the right hand side of equations (5) and (6) the presence of the terms  $(u_x t)$  and  $(u'_x t')$  respectively. We have to find out the names for them. Taking into account that clock  $C_0^0$  is located at the common origin of time at the common origins of the reference frames,  $x = u_x t$  can represent the space coordinate of clock  $C_0^0$  in the  $S$  frame when the synchronized clocks of that frame read  $t$  or the distance travelled by  $C_0^0$  during the time interval  $(t-0)$ , measured of course in  $S$ . For the same reasons,  $x' = u'_x t'$  can represent the distance travelled by  $C_0^0$  in the time interval  $(t'-0)$ , measured in  $S'$  or the space coordinate of clock  $C_0^0$  in the  $S'$  frame when the synchronized clocks read their  $t'$ . With this new notation, equations (5) and (6) become

$$t = \gamma_v \left( t' + \frac{v x'}{c^2} \right); \quad \gamma_v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (7)$$

$$t' = \gamma_v \left( t - \frac{v x}{c^2} \right) \quad (8)$$

The space coordinates  $x$  and  $x'$  transform as

$$x = u_x t = \gamma_v t' (u'_x + v) = \gamma_v (x' + v t') \quad (9)$$

and

$$x' = u'_x t' = \gamma_v t (u_x - v) = \gamma_v (x - v t) . \quad (10)$$

We have derived so far the Lorentz transformations for the events  $E(x = u_x t, y = 0, t)$  associated with the fact that clock  $C_0^0$  meets clock  $C(x = u_x t, y = 0)$  and  $E'(x' = u'_x t', y' = 0, t')$  associated with the fact that clock  $C_0^0$  meets clock  $C'(x' = u'_x t', y' = 0)$ . Because the two events take place at the same point in space, relativists say that they represent the same event.

### 3. Relativistic Dynamics

Consider that a particle is located at rest at the origin  $O^0$  of  $S^0$ . Observers in that frame measure its rest mass  $m^0$ . Using the addition law of relativistic velocities it was shown<sup>8</sup> that as measured from  $S$  the mass of the particle is

$$m = \frac{m_0}{\sqrt{1 - \frac{u_x^2}{c^2}}} , \quad (11)$$

whereas as measured from  $S'$  it is

$$m' = \frac{m_0}{\sqrt{1 - \frac{u_x'^2}{c^2}}}, \quad (12)$$

resulting that  $m$  and  $m'$  are related by

$$m = m' \frac{\sqrt{1 - \frac{u_x'^2}{c^2}}}{\sqrt{1 - \frac{u_x^2}{c^2}}} = m' \frac{1 + \frac{vu_x'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma_v \left( m' + \frac{v(m'u_x')}{c^2} \right), \quad (13)$$

and by

$$m' = m \frac{1 - \frac{vu_x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma_v \left( m - \frac{v(mu_x)}{c^2} \right). \quad (14)$$

It is considered that  $m$  and  $m'$  represents the relativistic mass of the same particle as measured from  $S$  and  $S'$  respectively. We detect in the right hand side of equations (13) and (14) the presence of the terms  $mu_x$  and  $m'u'_x$  which have the physical dimensions of momentum. Physicists baptize physical quantities by a single name and label them by a single notation. So they introduce the notations

$$p_x = mu_x = \frac{m^0 u_x}{\sqrt{1 - \frac{u_x^2}{c^2}}} \quad (15)$$

$$p'_x = m'u'_x = \frac{m^0 u'_x}{\sqrt{1 - \frac{u_x'^2}{c^2}}} \quad (16)$$

$p_x$  and  $p'_x$  representing the relativistic momentum of the particle with rest mass  $m^0$  as measured from  $S$  and  $S'$  respectively. The relativistic momentum transforms as

$$p_x = mu_x = \gamma_v m'(u'_x + v) = \gamma_v (p'_x + vm') \quad (17)$$

$$p'_x = m'u'_x = \gamma_v m(u_x - v) = \gamma_v (p_x - vm). \quad (18)$$

The physical quantity  $E = mc^2$  ( $E' = m'c^2$ ) has the physical dimensions of energy. Due to the invariance of  $c$ , it transform as mass does

$$E = E' \gamma_v \left( 1 + \frac{vu'_x}{c^2} \right) = \gamma_v (E' + vp'_x) \quad (19)$$

and

$$E' = E\gamma_v \left( 1 - \frac{vu_x}{c^2} \right) = \gamma_v (E - vp_x). \quad (20)$$

Expressed as a function of energy, equations (19) and (20) become

$$p_x = \gamma_v \left( p'_x + \frac{vE'}{c^2} \right) \quad (21)$$

and

$$p'_x = \gamma_v \left( p_x + \frac{vE}{c^2} \right). \quad (22)$$

Equations (19) and (22) put the bane on the concept of relativistic mass as many physicists require<sup>9</sup>.

A remarkable property of the relativistic energy consists in the fact that a particle at rest is characterized by the rest energy

$$E^0 = m^0 c^2. \quad (23)$$

Because the single supplementary energy a free particle moving with constant velocity can possess is its kinetic energy  $E_k$ , we have for it

$$E_k = E - E_0 = E_0(\gamma_v - 1) = m^0 c^2(\gamma_v - 1). \quad (24)$$

It is clear that  $p_x(p'_x)$ ,  $E (E')$  express the momentum and the energy of the same particle as measured from  $S$  and  $S'$ . The traditional way to derive the equations presented above is to consider a collision between two particles from two inertial reference frames and imposing that momentum and energy conserve<sup>10</sup>.

The truth of equation (11) is confirmed by the experiments performed Bucherer<sup>11</sup> with electrons moving in a magnetic field, which obtained an excellent agreement working with it. This was and continues to be a great proof in favour of the special theory of relativity in all its fields, all being the consequence of Einstein's postulate of relativity. It is considered<sup>11</sup> that to this date no other explanation of Bucherer's experiment is known.

#### 4. Relativistic electrodynamics

We consider now a capacitor  $C_0$  the plates of which are parallel to the  $X^0 O^0 Z^0$  plane, at rest in  $S^0$  and charged with electrical charges  $Q$  and  $-Q$  respectively. It creates an electric field  $E_y^0$ , oriented in the positive direction of the  $O^0 Y^0$  axis given by

$$E_y^0 = \frac{Q}{\varepsilon_0 \varepsilon_r A^0}, \quad (25)$$

$A^0$  representing the surface of one of the plates,  $\varepsilon_0$  and  $\varepsilon_r$  representing invariant electrical constants. The electrical neutrality of atoms convinces us that the electric charge is a relativistic invariant. When detected from  $S$ , the sides of the plate parallel with the overlapped axes undergo a Lorentz contraction and so the electric field measured from  $S$  is

$$E_y = \frac{Q}{\varepsilon_0 \varepsilon_r A^0 \sqrt{1 - \frac{u_x^2}{c^2}}} \quad (26)$$

whereas in  $S'$  it is

$$E'_y = \frac{Q}{\epsilon_0 \epsilon_r A^0 \sqrt{1 - \frac{u_x^2}{c^2}}} \quad (27)$$

The same procedure, as in the previous cases leads to the following relationship between  $E_y$  and  $E'_y$

$$E_y = E'_y \gamma_v \left( 1 + \frac{v u'_x}{c^2} \right) = \gamma_v \left( E'_y + \frac{v(u'_x E'_y)}{c^2} \right) \quad (28)$$

$$E'_y = E_y \gamma_v \left( 1 - \frac{v u_x}{c^2} \right) = \gamma_v \left( E_y - \frac{v(u_x E_y)}{c^2} \right) \quad (29)$$

We detect in the right hand side of equations (28) and (29) the presence of the terms  $(u_x E_y)$  and  $(u'_x E'_y)$  and label them  $B_z$  and  $B'_z$  respectively, representing the  $OZ(OZ')$  components of a vector  $\vec{B}(0,0,B)$  in  $S$  and  $\vec{B}'(0,0,B')$  in  $S'$ , baptized magnetic field. Its direction is chosen in such a way that the vector products  $\vec{u}_x \times \vec{E}_y$  and  $\vec{u}'_x \times \vec{E}'_y$  show in the same direction as  $E_y$  and  $E'_y$  do, respectively. It transform as

$$B_z = \gamma_v E'_y (u'_x + v) = \gamma_v (B'_z + v E'_y) \quad (30)$$

$$B'_z = \gamma_v E_y (u_x - v) = \gamma_v (B_z - v E_y) \quad (31)$$

Expressed as a function of the magnetic field, equations (28) and (29) become

$$E_y = \gamma_v \left( E'_y + \frac{v B'_z}{c^2} \right) \quad (32)$$

$$E'_y = \gamma_v \left( E_y - \frac{v B_z}{c^2} \right) \quad (33)$$

With the plates of the capacitor parallel to the  $Y^0 O^0 Z^0$ , it creates an electric field parallel to the overlapped axes. In that case the sides of the plates are not subjected to the Lorentz contraction and so

$$E_x^0 = E_x = E'_x \quad (34)$$

The transformation equations for the components  $E_z(E'_z)$  and  $B_y(B'_y)$  are obtained starting with the capacitor the plates of which are parallel to the  $XOY$  plane producing an electric field orientated along the  $OZ$  axis. The traditional way to derive the equations presented above is to use the transformation properties of force<sup>12,13</sup>.

## 5. Conclusions

We have shown that if a proper physical quantity  $F^0$  (proper time, proper mass, proper electric field) can be defined in a given reference frame  $S^0$  which moves with velocity  $u_x$  relative to  $S$  and with velocity  $u'_x$  relative to  $S'$  and transforms as

$$F = \frac{F^0}{\sqrt{1 - \frac{u_x^2}{c^2}}} \quad (35)$$

$$F' = \frac{F^0}{\sqrt{1 - \frac{u'^2_x}{c^2}}} \quad (36)$$

respectively, then a transformation equation relating  $F$  and  $F'$  can be derived using the addition law of velocities as a single relativistic ingredient. The transformation generate a new physical quantity ( $uF$ ), ( $u'F'$ ) in  $S$  and in  $S'$  respectively for which physicists find adequate names and the transformation of which involves the transformation of  $F$  and  $F'$  and the addition law of velocities.

As compared with the traditional derivations which are very laborious, our derivation is heuristic, being based on facts known from classical physics, uses simple algebra being time saving having a good relationship between time invested and understanding achieved. We consider that it convinces the reluctants about the truth of special relativity. It does not charge the memory of the teacher or of the learner, who in the traditional approaches, have to learn in each chapter of physics, a new scenario.

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