Journal of Theoretics
Volume 5-4, Aug-Sept 2003

A New Approach to the Field Theory

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Abstract: In this study, we suggest a new model for Lagrangian field theory. Later by means of this model we derive Maxwell and Dirac equations. Finally, we conclude that each interaction type in nature may be expressed by a certain space which we call it as interaction space.

Keywords: Lagrangian, Field Theory, Maxwell equations, Dirac Equations; Photons; Interaction Space.

1. Interaction Space Concept

Field theory tries to explain interactions of some particles with each other and properties of these interactions. There are different kind particles and these are represented by different type functions or briefly "fields"; such as scalar fields, vector fields, gauge fields, spinors. Particles interact with each other and cause four main types interactions, which are known as gravitation, electromagnetic, weak and strong. In order to explain these interactions and particles, lots of studies have been carried out and various theories have been set forth. However; sources of these particles and interactions have been not clarified systematically and satisfactorily so far.

In this paper, we would like to introduce a new way to look into origins of fields and interactions analytically, by adopting some classical mechanics ideas to field theory. In order to realize this; we will propose interaction space concept similar to the Cartesian coordinates in classical mechanics and we will try to get results of this assumption by some examples. When we investigate results of these examples we will be confronted with various long calculations. However all of them will be only some straightforward calculations, after our assumptions.

In classical mechanics there is a three-dimensional Euclidean space and any point in this space is showed by using position vector $\mathbf{r}$ (Fig.1.a). In Fig. 1.a x_i (x, y and z) are coordinate axes and $\mathbf{e}_i$ are unit vectors of this three-dimensional Euclidean space ($i = 1,2,3$). If the point is moving, x_i will be a function of the time (t). In order to explain motion of the point, $\mathbf{r}(t)$ or at least equations of motion which will lead to calculate $\mathbf{r}(t)$ must be designed. In classical mechanics; the motion of any point can be determined by means of a well-known scalar quantity which is known as Lagrangian. For any point in a classical system Lagrangian may be a function of coordinates, velocities and the time as

$$L = L(t) \equiv L(x_i(t), \dot{x}_i(t); t) \quad . \tag{1}$$

In this expression $\dot{x}_i(t) = \frac{dx_i}{dt}$ are velocities, x_i are Cartesian coordinates and t is the time.

Similar Lagrangian approach is of use in field theory and we will use it in this study frequently. For the most general case Lagrangian (density) of fields can be

$$\mathcal{L} = \mathcal{L}(x^\mu) \equiv \mathcal{L}\left(\partial_\mu \psi_a(x^\nu), \psi_a(x^\nu); x^\nu\right), \tag{2}$$

where $\psi_a(x^\mu)$ are fields, x^μ are space-time and $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. Also $\mu, \nu = 0,1,2,3$ and if there are N fields, $a = 0,1,2,....,(N-1)$.

It can be seen very easily by comparing Eq. (1) and (2); $\psi_a(x^\mu)$ have similar role with $x_i(t)$. Also there is the same relation between the time (t) and the space-time (x^μ). This point is the first step for interaction space model. Since $x_i(t)$ are coordinates of a three-dimensional Euclidean space; we can expect and propose that $\psi_a(x^\mu)$ *are coordinates of a new N-dimensional Euclidean space for the most general case* as a result of resemblance between $\psi_a(x^\mu)$ and $x_i(t)$. In addition, similar to classical mechanics we assume that there must be some unit vectors in this space. So any point in this space, can be shown by these coordinates and unit vectors, like classical mechanics. To be exact, we suppose that any point in this new space can be shown by a position vector as

$$\mathbf{R} = \psi_a \mathbf{A}_a \, ,$$

where $\mathbf{R}$ is the position vector of the point and $\mathbf{A}_a$ are unit vectors of this new space ($a = 0,1,2,....,(N-1)$).

Similar to classical case; unit vectors satisfy the dot and cross product relations as

$$\mathbf{A}_a \bullet \mathbf{A}_b = \delta_{ab} \, , \tag{3}$$

$$\mathbf{A}_a \mathbf{x} \mathbf{A}_b = \in_{abc} \mathbf{A}_c \, . \tag{4}$$

Thus; we have constructed a new Euclidean coordinate system. Hereafter; we will call it as *interaction space*. Fig. 2.a is symbolic figure of a four-dimensional interaction space.

The motion of a point in classical mechanics, depends on only one main parameter and it is the time (t). However; in the interaction space; the motion point will be a function of the space-time (x^μ). That is, there are four major parameters for the motion. Therefore, in order to describe motion of any point in interaction space we need to determine position vector of the point as a function of the space-time as

$$\mathbf{R} = \mathbf{R}(x^\mu)$$

or at least, equations of motion of the interaction space coordinates must be established. In order to find equations of motion, Lagrangian of the point must be determined. In this study; we will focus on only motion of a free point in interaction space. For this point; we will try to determine Lagrangian by using similarities between classical mechanics and field theory. In classical mechanics Lagrangian of a free object is $L(t) = \frac{m}{2}\left[\left(\frac{d\mathbf{r}}{dt}\right) \bullet \left(\frac{d\mathbf{r}}{dt}\right)\right]$ where m is mass of object. From similarity between classical mechanics and field theory, we assume

$$\mathcal{L}=\frac{M}{2}\left(\partial_{\mu}\mathbf{R}^{*}\right)\bullet\left(\partial_{\mu}\mathbf{R}\right) \tag{5}$$

parallel to classical mechanics, where M is a constant, $\mathbf{R}^{*}$ is the complex conjugate of $\mathbf{R}$.

After Lagrangian of the system is determined, we write an action integral

$$S=\int\mathcal{L}(x^{\mu})dx^{4}$$

and apply the condition $\delta S=0$ at the endpoints of motion, we reach Euler-Lagrange equations

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\psi_{a}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\psi_{a}}=0 .$$

Using these equations, equations of the motion can be calculated easily.

2. Applications of the Interaction Space

In this section, we present two examples for motion of a free point in the interaction space. For the both applications, we assume interaction spaces are 4-dimensional and their unit vectors are real, since results of these are easier to understand and explain. However, there will be a small difference between these two examples; the interaction space of the first example is fixed but the second interaction space rotates with respect to any other fixed space. This difference will cause very interesting results.

For the first example since interaction space is fixed, the position vector of any moving point in this space will be

$$\mathbf{R}(x^{\alpha})=\psi_{\mu}(x^{\alpha})\mathbf{A}_{\mu}, \tag{6}$$

and the complex conjugate of Eq. (6) is

$$\mathbf{R}^{*}(x^{\alpha})=\psi^{\mu}(x^{\alpha})\mathbf{A}_{\mu},$$

where and in future upper indices on interaction space quantities denote complex conjugate of them. That is

$$\psi^{\mu}\equiv\left(\psi_{\mu}\right)^{*} .$$

Eqs. (3) and (5) give

$$\mathcal{L}=\frac{M}{2}\left(\partial_{\mu}\psi^{\nu}\right)\left(\partial_{\mu}\psi_{\nu}\right) . \tag{7}$$

Eq. (7) leads to following equations of motion

$$\begin{aligned}\Box\psi^{\mu}&=0,\\ \Box\psi_{\mu}&=0\end{aligned} , \tag{8}$$

where

$$\Box\equiv\partial_{\mu}\partial_{\mu}\equiv-\frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}+\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}+\frac{\partial^{2}}{\partial z^{2}} .$$

We see that Eq. (8) represents some massless particles. We can derive a conserved quantity from Eqs. (8) as

$$J_{\mu}=\psi^{\nu}\left(\partial_{\mu}\psi_{\nu}\right)-\psi_{\nu}\left(\partial_{\mu}\psi^{\nu}\right) . \tag{9}$$

The next example is on a 4-dimensional interaction space again. But this time, it is not a fixed space and it rotates with respect to a fixed space (Fig. 2.b). Because of this, unit vectors will be observed as if some functions of the space-time by an observer in a fixed interaction space and the position vector of any point will be

$$\mathbf{R}(x^{\mu}) = \psi_{\nu}(x^{\mu})\mathbf{A}_{\nu}(x^{\mu}),$$

where $\mathbf{A}_{\mu}$ are unit vectors of the interaction space.

Since calculation of Lagrangian includes long mathematical course of actions, detailed information and calculation about Lagrangian for this 4-dimensional interaction space are given in the appendix A section. After time-consuming calculations from Eq. (A.17) the Lagrangian is

$$\mathcal{L}\left(x^{\mu}\right) = \frac{M}{2}\left[\left(\partial_{\nu}\psi^{\mu}\right)\left(\partial_{\nu}\psi_{\mu}\right) - \frac{2}{36}\psi^{\alpha}\psi_{\alpha}\left(\mathbf{F}_{\nu\mu}\bullet\mathbf{F}_{\mu\nu}\right) + \frac{1}{36}\psi^{\mu}\psi_{\alpha}\in_{\alpha\lambda\eta}\in_{\mu\rho\gamma}\left(\mathbf{F}_{\lambda\eta}\bullet\mathbf{F}_{\gamma\rho}\right)\right], \tag{11}$$

where

$$\mathbf{F}_{\nu\mu} = \partial_{\nu}\mathbf{A}_{\mu} - \partial_{\mu}\mathbf{A}_{\nu}.$$

Lagrangian (11) gives following equations of motion

$$\partial_{\theta}\left[4\psi^{\alpha}\psi_{\alpha}\mathbf{F}_{\theta\lambda} - \in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\left(\psi^{\alpha}\psi_{\mu} + \psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right] = 0, \tag{12}$$

$$\frac{M}{2}\left[\Box\psi_{\lambda} + \frac{2}{36}\left(\mathbf{F}_{\mu\nu}\bullet\mathbf{F}_{\nu\mu}\right)\psi_{\lambda}\right] = \frac{M}{2}\left[\frac{1}{36}\in_{\alpha\mu\eta}\in_{\lambda\rho\gamma}\left(\mathbf{F}_{\mu\eta}\bullet\mathbf{F}_{\gamma\rho}\right)\psi_{\alpha}\right], \tag{13}$$

$$\frac{M}{2}\left[\Box\psi^{\lambda} + \frac{2}{36}\left(\mathbf{F}_{\mu\nu}\bullet\mathbf{F}_{\nu\mu}\right)\psi^{\lambda}\right] = \frac{M}{2}\left[\frac{1}{36}\in_{\alpha\mu\eta}\in_{\lambda\rho\gamma}\left(\mathbf{F}_{\mu\eta}\bullet\mathbf{F}_{\gamma\rho}\right)\psi^{\alpha}\right]. \tag{14}$$

In the last two equations, we have not eliminated $\frac{M}{2}$ since we will use it later. Although these equations seem unfamiliar, we can show they may be more general form of Maxwell and Dirac equations.

Firstly, Eq. (12) can be written as

$$\partial_{\theta}\mathbf{F}_{\theta\lambda} = \mathbf{K}_{\lambda}, \tag{15}$$

where

$$\mathbf{K}_{\lambda} = \frac{1}{4R^{2}}\partial_{\theta}\left[\in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\left(\psi^{\alpha}\psi_{\mu} + \psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right] - \frac{2}{R}\left(\partial_{\theta}R\right)\mathbf{F}_{\theta\lambda}, \tag{16}$$

where

$$R^{2} = \psi^{\mu}\psi_{\mu}.$$

$\mathbf{K}_{\lambda}$ is a conserved quantity (It is proved in the appendix B section.), therefore it satisfies

$$\partial_{\lambda}\mathbf{K}_{\lambda} = 0.$$

Then Eq. (15) is nothing but Maxwell equations in a different form. The well-known Maxwell equations can be derived by writing definition of $\mathbf{F}_{\theta\lambda}$ in terms $\mathbf{A}_{\lambda}$ and by writing components of the electric and magnetic fields vectors **E** and **B** as

$$E_{1} = \mathbf{F}_{01} = \partial_{0}\mathbf{A}_{1} - \partial_{1}\mathbf{A}_{0}, \qquad E_{2} = \mathbf{F}_{02} = \partial_{0}\mathbf{A}_{2} - \partial_{2}\mathbf{A}_{0}, \qquad E_{3} = \mathbf{F}_{03} = \partial_{0}\mathbf{A}_{3} - \partial_{3}\mathbf{A}_{0},$$

$$B_{1} = \mathbf{F}_{23} = \partial_{2}\mathbf{A}_{3} - \partial_{3}\mathbf{A}_{2}, \qquad B_{2} = \mathbf{F}_{31} = \partial_{3}\mathbf{A}_{1} - \partial_{1}\mathbf{A}_{3}, \qquad B_{3} = \mathbf{F}_{12} = \partial_{1}\mathbf{A}_{2} - \partial_{2}\mathbf{A}_{1}.$$

Here $E_{1}, E_{2}, E_{3}, B_{1}, B_{2}$ and B_{3} are components of the electric and magnetic fields vectors **E** and **B** respectively.

Since Eq. (15) is Maxwell equations and they describe photons, Eqs. (13) and (14) should give us an idea about electrons. We know electrons are represented by Dirac spinors and Dirac equations. Subsequently, we encounter an interesting situation; Eqs. (13) and (14) should be Dirac equations in a different and more general form. We show that Eqs. (13) and (14) can be reduced Dirac equations, in the appendix C section.

If Eq. (15) is Maxwell's equations, $\mathbf{K}_\lambda$ in (16) must be electrical charge-current density. However, we can derive another conserved quantity from Eqs. (13) and (14) as

$$J_\mu = \psi^\alpha \left(\partial_\mu \psi_\alpha\right) - \psi_\alpha \left(\partial_\mu \psi^\alpha\right) \quad . \tag{17}$$

Two different conserved quantities are not usual situation for electrodynamics. However; we know different type densities from weak and strong interactions. Explanation of this situation is out of scope of this study.

3. Conclusion

We can give several results of this study but they will be extremely speculative at this point. Because of this fact, we leave interpreting of these results to another study and summarize the most important feature of this paper. We have presented interaction space idea for field theory and using this idea we have succeeded to derive Maxwell equations and more general forms of Dirac equations, at least mathematically. Maxwell and Dirac equations are basic equations of electrodynamics/electromagnetism and all electromagnetic interactions could be explained by these equations. This indicates we have succeeded to explain all electromagnetic interactions by using a four dimensional Euclidean space. If electromagnetic interactions can be explained by a certain interaction space, an important question arises; "**Can we use different type interaction spaces to describe other interactions?**" If the answer is positive; it is very clear that interaction space assumption shows the newest way to inspect over again field theory, fields, particles and interactions.

Appendix A

In this appendix, we derive Lagrangian of a free point in a 4-dimensional interaction space which rotates with respect to a fixed space. For this 4-dimensional interaction space position vector of the point is

$$\mathbf{R} = \mathbf{R}(x^\mu) = \psi_a(x^\mu)\mathbf{A}_a(x^\mu) \, ,$$

$$\partial_\nu \mathbf{R} = \partial_\nu \psi_\mu \mathbf{A}_\mu + \psi_\mu \partial_\nu \mathbf{A}_\mu \, .$$

Since $\mathbf{A}_\mu$ are unit vectors of the interaction space, $d\mathbf{A}_\mu$ must be perpendicular to $\mathbf{A}_\mu$ and it must lie in the plane of the other unit vectors. Because of this we can write for $d\mathbf{A}_\mu$

$$d\mathbf{A}_\mu = \Pi_{\nu\mu\eta} dx^\nu \mathbf{A}_\eta \, , \tag{A.1}$$

where $\Pi_{\nu\mu\eta}$ are some coefficients and they are very important quantities. From the last equation we write

$$\partial_\nu \mathbf{A}_\mu = \Pi_{\nu\mu\eta} \mathbf{A}_\eta \tag{A.2}$$

and

$$\partial_\nu \mathbf{R} = \partial_\nu \psi_\mu \mathbf{A}_\mu + \psi_\mu \Pi_{\nu\mu\eta} \mathbf{A}_\eta \, .$$

Assume that $\Pi_{\nu\mu\eta}$ can be written as

$$\Pi_{\nu\mu\eta} = \Omega_{\nu\alpha} \in_{\alpha\mu\eta} \tag{A.3}$$

$$\partial_\nu \mathbf{R} = \partial_\nu \psi_\mu \mathbf{A}_\mu + \psi_\mu \Omega_{\nu\alpha} \in_{\alpha\mu\eta} \mathbf{A}_\eta \, .$$

Using Eq. (4) we can write the last equation as

$$\partial_\nu \mathbf{R} = \partial_\nu \psi_\mu \mathbf{A}_\mu + \psi_\mu \Omega_{\nu\alpha} \mathbf{A}_\alpha \mathbf{x} \mathbf{A}_\mu \, ,$$

or

$$\partial_\nu \mathbf{R} = \partial_\nu \psi_\mu \mathbf{A}_\mu + \left(\Omega_{\nu\alpha}\mathbf{A}_\alpha\right)\mathbf{x}\left(\psi_\mu \mathbf{A}_\mu\right).$$

Let

$$\mathbf{\Omega}_\nu = \Omega_{\nu\alpha}\mathbf{A}_\alpha .$$

Thus

$$\partial_\nu \mathbf{R} = \partial_\nu \psi_\mu \mathbf{A}_\mu + \mathbf{\Omega}_\nu \mathbf{x}\mathbf{R} .$$

Derivative of the unit vectors leads us to define a new vector $\mathbf{\Omega}_\nu$. It has similar function with the angular velocity vector $\boldsymbol{\omega}$ in classical mechanics. Therefore, derivatives of the position vector can be written as

$$\left(\partial_\nu \mathbf{R}\right)_{fixed} = \left(\partial_\nu \mathbf{R}\right)_{relative} + \mathbf{\Omega}_\nu \mathbf{x}\mathbf{R}$$

In the last expression, $\left(\partial_\nu \mathbf{R}\right)_{fixed}$ and $\left(\partial_\nu \mathbf{R}\right)_{relative}$ denote derivatives of **R** with respect to fixed and relative interaction spaces, respectively.

So we write position vector in terms of $\mathbf{\Omega}_\nu$ as

$$\partial_\nu \mathbf{R} = \partial_\nu \psi_\mu \mathbf{A}_\mu + \psi_\mu \Omega_{\nu\alpha} \in_{\alpha\mu\eta} \mathbf{A}_\eta ,$$

$$\partial_\nu \mathbf{R}^* = \partial_\nu \psi^\lambda \mathbf{A}_\lambda + \psi^\lambda \Omega_{\nu\beta} \in_{\beta\lambda\gamma} \mathbf{A}_\gamma .$$

Lagrangian of these can be found as

$$\mathcal{L} = \frac{M}{2}\left(\partial_\nu \mathbf{R}^*\right)\bullet\left(\partial_\nu \mathbf{R}\right) = \frac{M}{2}\left(\partial_\nu \psi^\lambda \mathbf{A}_\lambda + \psi^\lambda \Omega_{\nu\beta} \in_{\beta\lambda\gamma} \mathbf{A}_\gamma\right)\bullet\left(\partial_\nu \psi_\mu \mathbf{A}_\mu + \psi_\mu \Omega_{\nu\alpha} \in_{\alpha\mu\eta} \mathbf{A}_\eta\right)$$

or as a result of Eq. (3)

$$\mathcal{L} = \frac{M}{2}\left[\partial_\nu \psi^\mu \partial_\nu \psi_\mu + \Omega_{\nu\alpha} \in_{\alpha\mu\eta} \left(\partial_\nu \psi^\eta \psi_\mu + \psi^\mu \partial_\nu \psi_\eta\right) + \psi^\lambda \psi_\mu \Omega_{\nu\alpha} \Omega_{\nu\alpha} \in_{\alpha\lambda\eta} \in_{\alpha\mu\eta}\right]. \qquad \text{(A.4)}$$

The second term in the parentheses of Eq. (A.4) vanishes as result of anti-symmetrical features of Levi Civita symbols. Lagrangian turns into

$$\mathcal{L} = \frac{M}{2}\left(\partial_\nu \psi^\mu \partial_\nu \psi_\mu + \psi^\lambda \psi_\mu \Omega_{\nu\alpha} \Omega_{\nu\alpha} \in_{\alpha\lambda\eta} \in_{\alpha\mu\eta}\right),$$

by performing summations over Levi Civita symbols Lagrangian becomes

$$\mathcal{L} = \frac{M}{2}\left(\partial_\nu \psi^\mu \partial_\nu \psi_\mu + \psi^\mu \psi_\mu \Omega_{\nu\alpha} \Omega_{\nu\alpha} - \psi^\mu \psi_\alpha \Omega_{\nu\alpha} \Omega_{\nu\mu}\right).$$

And from Eq. (A.3)

$$\Omega_{\nu\alpha} = \frac{1}{3} \in_{\alpha\mu\eta} \Pi_{\nu\mu\eta} .$$

Using the last equation we can write Lagrangian as

$$\mathcal{L} = \frac{M}{2}\left(\partial_\nu \psi^\mu \partial_\nu \psi_\mu + \frac{2}{9}\psi^\alpha \psi_\alpha \Pi_{\nu\mu\eta}\Pi_{\nu\mu\eta} - \frac{1}{9}\psi^\mu \psi_\alpha \in_{\alpha\lambda\eta} \in_{\mu\rho\gamma} \Pi_{\nu\lambda\eta}\Pi_{\nu\rho\gamma}\right). \qquad \text{(A.5)}$$

We can simplify Eq. (A.5) by writing $\Pi_{\nu\mu\eta}$'s in terms of unit vectors. For this we must turn unit vectors. By taking differential of Eq. (3)

$$d\mathbf{A}_\mu \bullet \mathbf{A}_\nu + \mathbf{A}_\mu \bullet d\mathbf{A}_\nu = 0 \qquad \text{(A.6)}$$

can be found. By replacing Eq. (A.1) into Eq. (A.6) we find

$$\Pi_{\nu\mu\eta} + \Pi_{\nu\eta\mu} = 0 . \qquad \text{(A.7)}$$

Eq. (A.7) shows that $\Pi_{\nu\mu\eta}$ is anti-symmetric with respect to the last two indices. We can show $\Pi_{\nu\mu\eta}$ is anti-symmetric with respect to the first two indices also. For this, we will define a "$\otimes$" product and we multiply Eq. (A.1) by dx^{μ} using this $\otimes$ product. Since we do not know anything about interaction space quantities we will not make any comment about properties of this product for the present. We will try to determine feature of it later, after some calculations.

If we multiply Eq. (A.1) by dx^{μ} and take summation over μ, we find

$$d\mathbf{A}_{\mu} \otimes dx^{\mu} = \Pi_{\nu\mu\eta} dx^{\nu} \otimes dx^{\mu} \mathbf{A}_{\eta} . \tag{A.8}$$

We can rewrite Eq. (A.8), by interchanging μ and ν indices on the right-hand side of it. Now, we have

$$d\mathbf{A}_{\nu} \otimes dx^{\nu} = \Pi_{\mu\nu\eta} dx^{\mu} \otimes dx^{\nu} \mathbf{A}_{\eta} . \tag{A.9}$$

Eqs. (A.8) and (A.9) are the equivalent, so we write

$$\Pi_{\mu\nu\eta} dx^{\mu} \otimes dx^{\nu} = \Pi_{\nu\mu\eta} dx^{\nu} \otimes dx^{\mu} .$$

According to the last equation there are two possibilities for $\otimes$ product. It is either commutative or anti-commutative. If it is commutative, $\Pi_{\nu\mu\eta}$ is symmetric with respect to first two indices. But this alternative is impossible; because we have found that it was anti-symmetric with respect to last two indices. It can not be symmetric with respect to first two indices and anti-symmetric with respect to last two indices at the same time. Then $\otimes$ product is anti-commutative, consequently $\Pi_{\nu\mu\eta}$ is anti-symmetric with respect to first two indices. That is; $\Pi_{\nu\mu\eta}$ is totally anti-symmetric. So

$$\Pi_{\mu\nu\eta} = -\Pi_{\nu\mu\eta} ,$$

$$\Pi_{\mu\nu\eta} = -\Pi_{\mu\eta\nu} ,$$

$$\Pi_{\mu\nu\eta} = -\Pi_{\eta\nu\mu} . \tag{A.10}$$

From Eq. (A.2) and using Eq. (3) we obtain

$$\Pi_{\nu\mu\eta} = \left(\partial_{\nu}\mathbf{A}_{\mu}\right) \bullet \mathbf{A}_{\eta} . \tag{A.11}$$

Using Eq. (A.10) we write (A.11) as

$$\Pi_{\nu\mu\eta} = \frac{1}{2}\left(\partial_{\nu}\mathbf{A}_{\mu} - \partial_{\mu}\mathbf{A}_{\nu}\right) \bullet \mathbf{A}_{\eta} .$$

Let

$$\mathbf{F}_{\nu\mu} = \partial_{\nu}\mathbf{A}_{\mu} - \partial_{\mu}\mathbf{A}_{\nu} .$$

Thus

$$\Pi_{\nu\mu\eta} = \frac{1}{2}\mathbf{F}_{\nu\mu} \bullet \mathbf{A}_{\eta} \tag{A.12}$$

and from (A.2) we can write

$$\frac{1}{2}\mathbf{F}_{\nu\mu} = \Pi_{\nu\mu\eta}\mathbf{A}_{\eta} . \tag{A.13}$$

We can derive some important relations for $\Pi_{\nu\mu\eta}$. If we multiply Eq. (A.12) by $\mathbf{A}_{\eta}$ and take summation over indices, from Eq. (A.13) we obtain

$$\Pi_{\nu\mu\eta}\mathbf{A}_{\eta} = \frac{1}{2}\left(\mathbf{F}_{\nu\mu} \bullet \mathbf{A}_{\eta}\right)\mathbf{A}_{\eta} = \frac{1}{2}\mathbf{F}_{\nu\mu} . \tag{A.14}$$

The dot product of Eq. (A.14) with $\frac{1}{2}\mathbf{F}_{\alpha\beta}$ yields

$$\frac{1}{2}\Pi_{\nu\mu\eta}\mathbf{A}_{\eta}\bullet\mathbf{F}_{\alpha\beta}=\frac{1}{4}\left(\mathbf{F}_{\nu\mu}\bullet\mathbf{A}_{\eta}\right)\left(\mathbf{A}_{\eta}\bullet\mathbf{F}_{\alpha\beta}\right)=\frac{1}{4}\left(\mathbf{F}_{\nu\mu}\bullet\mathbf{F}_{\alpha\beta}\right)=\Pi_{\nu\mu\eta}\Pi_{\alpha\beta\eta}\,. \tag{A.15}$$

The last identity for $\Pi_{\nu\mu\eta}$ can be derived by using Eq. (A.10) as

$$\Pi_{\nu\mu\eta}\Pi_{\alpha\beta\eta}=\Pi_{\nu\eta\mu}\Pi_{\alpha\eta\beta}=\Pi_{\eta\mu\nu}\Pi_{\eta\beta\alpha}\,. \tag{A.16}$$

So using these equations we can find Lagrangian of a relative four-dimensional interaction space as

$$\mathcal{L}\left(x^{\mu}\right)=\frac{M}{2}\left[\left(\partial_{\nu}\psi^{\mu}\right)\left(\partial_{\nu}\psi_{\mu}\right)+\frac{2}{36}\psi^{\alpha}\psi_{\alpha}\left(\mathbf{F}_{\nu\mu}\bullet\mathbf{F}_{\nu\mu}\right)-\frac{1}{36}\psi^{\mu}\psi_{\alpha}\in_{\alpha\lambda\eta}\in_{\mu\rho\gamma}\left(\mathbf{F}_{\lambda\eta}\bullet\mathbf{F}_{\rho\gamma}\right)\right] \tag{A.17}$$

Appendix B

In this appendix, we will show that $\mathbf{K}_{\lambda}$ is a conserved quantity. We have found that

$$\partial_{\theta}\mathbf{F}_{\theta\lambda}=\mathbf{K}_{\lambda},$$

$$\mathbf{K}_{\lambda}=\frac{1}{4R^{2}}\in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\partial_{\theta}\left(\left(\psi^{\alpha}\psi_{\mu}+\psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right)-\frac{2}{R}\left(\partial_{\theta}R\right)\mathbf{F}_{\theta\lambda}\,, \tag{B.1}$$

where

$$R^{2}=\psi^{\mu}\psi_{\mu}\,.$$

By applying ∂_{λ} to Eq. (B.1) we have

$$\partial_{\lambda}\mathbf{K}_{\lambda}=-\frac{1}{2R^{3}}\left(\partial_{\lambda}R\right)\in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\partial_{\theta}\left(\left(\psi^{\alpha}\psi_{\mu}+\psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right)+\frac{1}{4R^{2}}\in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\partial_{\lambda}\partial_{\theta}\left(\left(\psi^{\alpha}\psi_{\mu}+\psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right)$$

$$+\frac{2}{R^{2}}\left(\partial_{\lambda}R\right)\left(\partial_{\theta}R\right)\mathbf{F}_{\theta\lambda}-\frac{2}{R}\left(\partial_{\lambda}\partial_{\theta}R\right)\mathbf{F}_{\theta\lambda}-\frac{2}{R}\left(\partial_{\theta}R\right)\left(\partial_{\lambda}\mathbf{F}_{\theta\lambda}\right).$$

As a result of summations over anti-symmetrical terms we have

$$\partial_{\lambda}\mathbf{K}_{\lambda}=-\frac{1}{2R^{3}}\left(\partial_{\lambda}R\right)\in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\partial_{\theta}\left(\left(\psi^{\alpha}\psi_{\mu}+\psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right)-\frac{2}{R}\left(\partial_{\theta}R\right)\left(\partial_{\lambda}\mathbf{F}_{\theta\lambda}\right).$$

We rewrite it as

$$\partial_{\lambda}\mathbf{K}_{\lambda}=-\frac{1}{2R^{3}}\left[4R^{2}\left(\partial_{\theta}R\right)\left(\partial_{\lambda}\mathbf{F}_{\theta\lambda}\right)+\left(\partial_{\lambda}R\right)\in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\partial_{\theta}\left(\left(\psi^{\alpha}\psi_{\mu}+\psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right)\right].$$

By interchanging λ and θ of the first term in the square parentheses we find

$$\partial_{\lambda}\mathbf{K}_{\lambda}=-\frac{1}{2R^{3}}\left(\partial_{\lambda}R\right)\left[4R^{2}\left(\partial_{\theta}\mathbf{F}_{\lambda\theta}\right)+\in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\partial_{\theta}\left(\left(\psi^{\alpha}\psi_{\mu}+\psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right)\right]$$

or

$$\partial_{\lambda}\mathbf{K}_{\lambda}=\frac{1}{2R^{3}}\left(\partial_{\lambda}R\right)\left[4R^{2}\left(\partial_{\theta}\mathbf{F}_{\theta\lambda}\right)-\in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\partial_{\theta}\left(\left(\psi^{\alpha}\psi_{\mu}+\psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right)\right]$$

or

$$\partial_{\lambda}\mathbf{K}_{\lambda}=\frac{1}{2R^{3}}\left(\partial_{\lambda}R\right)\left\{\partial_{\theta}\left[4R^{2}\mathbf{F}_{\theta\lambda}-\in_{\alpha\theta\lambda}\in_{\mu\rho\eta}\left(\left(\psi^{\alpha}\psi_{\mu}+\psi^{\mu}\psi_{\alpha}\right)\mathbf{F}_{\rho\eta}\right)\right]-8R\left(\partial_{\theta}R\right)\mathbf{F}_{\theta\lambda}\right\}.$$

From Eq. (12) terms in square brackets is identically zero. So

$$\partial_{\lambda}\mathbf{K}_{\lambda}=-\frac{4}{R^{2}}\left(\partial_{\lambda}R\right)\left(\partial_{\theta}R\right)\mathbf{F}_{\theta\lambda}\equiv 0\,. \tag{B.2}$$

From Eq. (B.2), we see that; since $\mathbf{F}_{\mu\nu}$ is anti-commutative, $\mathbf{K}_\lambda$ must be conserved as a result of mathematical requirement.

Appendix C

In this appendix, we will show how Eqs. (13) and (14) can be reduced to Dirac equations. But first, in quantum mechanics a wave function satisfy

$$\mathcal{H}\Psi = \mathcal{E}\Psi\,, \tag{C.1}$$

where $\mathcal{H}$ is Hamiltonian operator, $\mathcal{E}$ is the energy operator and Ψ is the wave function. Similarly we write

$$\mathcal{H}^2\Psi = \mathcal{E}^2\Psi\,, \tag{C.2}$$

or

$$\left(\mathcal{H}^2 - \mathcal{E}^2\right)\Psi = 0 \tag{C.3}$$

By writing $\mathcal{E} = \mathbf{i}\hbar\frac{\partial}{\partial t}$, $\mathcal{H}^2 = c^2p^2 + c^4m_0^2$ and $p = -\mathbf{i}\hbar\nabla$, where c is speed of the light, p is the momentum, $\hbar$ the Planck constant and m_0 is rest mass of the particle. Thus (C.3) is

$$c^2\hbar^2\left(\partial_\mu\partial_\mu + \left(\frac{cm_0}{\hbar}\right)^2\right)\Psi = 0 \tag{C.4}$$

If we show that Eq. (C.4) can be identical with Eq. (13), we will have proved that Eqs. (13) and (14) are more general form of Dirac equations. Because Dirac equations can be derived from Eq. (C.4), by writing operators in terms of some matrices as Dirac did. So; if Eqs. (C.4) and (13) are identical we can derive Dirac equations from Eq. (13) also. Eq. (13) can be written as

$$\frac{M}{2}\left[\partial_\mu\partial_\mu + \frac{2}{36}\left(\mathbf{F}_{\mu\nu}\bullet\mathbf{F}_{\nu\mu}\right)\right]\psi_\lambda = \frac{M}{2}\left[\left(\frac{1}{6}\in_{\lambda\rho\gamma}\mathbf{F}_{\gamma\rho}\right)\bullet\left(\frac{1}{6}\in_{\alpha\mu\eta}\mathbf{F}_{\mu\eta}\psi_\alpha\right)\right]. \tag{C.5}$$

In fact Eq. (C.5) shows equations of four different functions. By defining an operator and

$$\mathcal{N}_\mu \equiv \frac{1}{6}\in_{\mu\alpha\beta}\mathbf{F}_{\alpha\beta}$$

and by letting $\Psi = \begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$ and $\mathcal{N} \equiv \begin{pmatrix}\mathcal{N}_0\\ \mathcal{N}_1\\ \mathcal{N}_2\\ \mathcal{N}_3\end{pmatrix}$, we can see Eq. (C.5) can be written in a simpler form as

$$\frac{M}{2}\left[\partial_\mu\partial_\mu + \frac{2}{36}\left(\mathbf{F}_{\mu\nu}\bullet\mathbf{F}_{\nu\mu}\right)\right]\Psi = \frac{M}{2}\mathcal{N}^\dagger\mathcal{N}\Psi\,,$$

Then, the last equation can be written

$$\frac{M}{2}\left[\partial_\mu\partial_\mu + \frac{2}{36}\left(\mathbf{F}_{\mu\nu}\bullet\mathbf{F}_{\nu\mu}\right) - \frac{1}{36}\in_{\alpha\lambda\eta}\in_{\mu\rho\gamma}\left(\mathbf{F}_{\lambda\eta}\bullet\mathbf{F}_{\gamma\rho}\right)\right]\Psi = 0\,. \tag{C.6}$$

Thus we have written Eq. (13) in a simple form by using matrix notation. Ψ in Eq. (C.6)) is a column matrix and Dirac equations contain similar wave function which name is spinor. If Eq. (13) can transform to Dirac equations, Ψ in Eq. (C.6)) must be the same with the function in Eq. (C.4). This condition satisfied only if operators of Eq. (C.4) and Eq. (C.6) are identical as

$$\frac{M}{2}\left[\partial_\mu\partial_\mu+\frac{2}{36}\left(\mathbf{F}_{\mu\nu}\bullet\mathbf{F}_{\nu\mu}\right)-\frac{1}{36}\in_{\alpha\lambda\eta}\in_{\mu\rho\gamma}\left(\mathbf{F}_{\lambda\eta}\bullet\mathbf{F}_{\rho\gamma}\right)\right]\equiv c^2\hbar^2\left(\partial_\mu\partial_\mu+\left(\frac{cm_0}{\hbar}\right)^2\right) \tag{C.7}$$

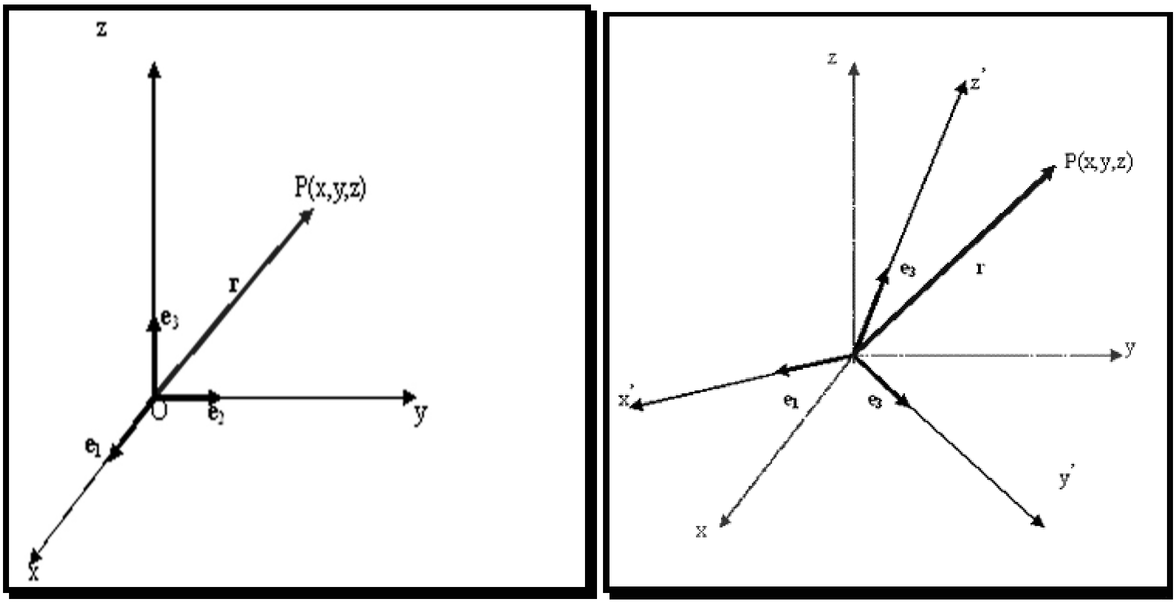
Eq. (C.7) requires

$$M=2c^2\hbar^2\,, \tag{C.8}$$

$$\frac{2}{36}\left(\mathbf{F}_{\mu\nu}\bullet\mathbf{F}_{\nu\mu}\right)-\frac{1}{36}\in_{\alpha\lambda\eta}\in_{\mu\rho\gamma}\left(\mathbf{F}_{\lambda\eta}\bullet\mathbf{F}_{\gamma\rho}\right)=\left(\frac{cm_0}{\hbar}\right)^2\,. \tag{C.9}$$

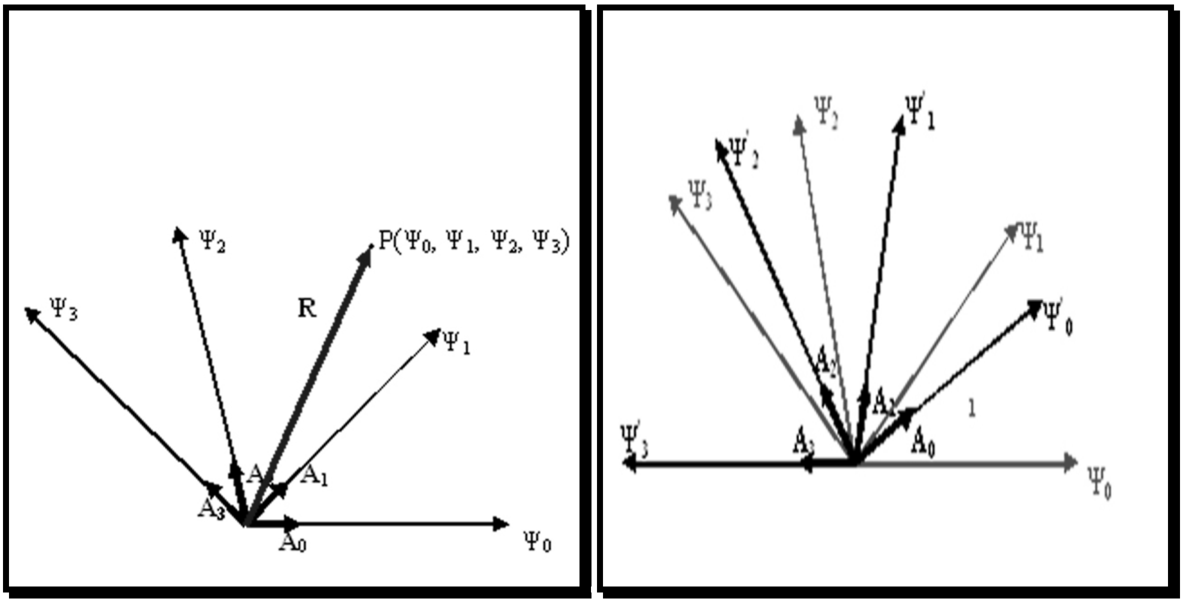
Eqs. (C.8) and (C.9) will make equivalence of Eq. (C.4) and Eq. (C.6) definite. Thus Ψ in Eq. (C.6) will be the same matrix and Eq. (13) will be identical with Eq. (C.4). After that we will be able to derive Dirac equations from Eq. (13). However, since $\mathbf{F}_{\mu\nu}$ may not be constant for the most general case, m_0 will not be constant always. That is; constant m_0 is possible only if $\mathbf{F}_{\mu\nu}$ or at least $\left(\mathbf{F}_{\mu\nu}\bullet\mathbf{F}_{\alpha\beta}\right)$ is constant.

Figures



(a) **(b)**

Figure 1. 3-dimensional Euclidean Space, in (a) **r** is the position vector of any arbitrary point of this space(**r**= x $\mathbf{e}_1$+y $\mathbf{e}_2$+z $\mathbf{e}_3$). (b) shows relative motion of a 3-dimensional Euclidean Space. Primed axes belong to the relative space.



(a) **(b)**

Figure 2. Symbolic figure of a 4-dimensional (Euclidean) interaction space. In (a) **R** is the position vector of any arbitrary point in this space ($\mathbf{R} = \psi_0\mathbf{A}_0 + \psi_1\mathbf{A}_1 + \psi_2\mathbf{A}_2 + \psi_3\mathbf{A}_3$). (b) shows relative motion of interaction space. Primed axes belong to the relative 4-dimensional Euclidean space.

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Received August 2002

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