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Extension of Complex Number by Mixed Number Algebra

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Abstract: Complex numbers are an extension of the real number system and can be written in the form a + bi, where a and b are both real numbers and i = √(- 1). The quaternions are just an extension of this complex number form. It was observed that Mixed Numbers are also the extension of complex numbers.
Keywords: quaternions, mixed numbers.

1. Extension of Complex Numbers

(a) By Quaternions

A quaternion is usually written as[1,2,3]

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

where a, b, c and d are scalar values and **i**, **j** and **k** are unique quaternions with the properties that[4]

$$\mathbf{i}^2 = -1,\ \mathbf{j}^2 = -1,\ \mathbf{k}^2 = -1$$

and

$$\mathbf{ij} = \mathbf{k},\ \mathbf{jk} = \mathbf{i},\ \mathbf{ki} = \mathbf{j},\ \mathbf{ji} = -\mathbf{i},\ \mathbf{kj} = -\mathbf{i},\ \mathbf{ik} = -\mathbf{j}\quad .$$

This is clearly an extension of the complex number system, where the complex numbers are those quaternions that have c = d = 0 and the real numbers are those that have b = c = d = 0.

The addition of two quaternions $q_1 = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $q_2 = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ is

$$q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)\,\mathbf{i} + (c_1 + c_2)\,\mathbf{j} + (d_1 + d_2)\,\mathbf{k} \quad \text{....................} \quad (1)$$

let $a_1 = a,\ b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k} = \mathbf{A},\ a_2 = b$ and $b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k} = \mathbf{B}$

then $q_1 = a + \mathbf{A}$ and $q_2 = b + \mathbf{B}$.

The product of these two quaternions is

$$q_1 q_2 = (a + \mathbf{A})(b + \mathbf{B}) = = ab - \mathbf{A}.\mathbf{B} + a\mathbf{B} + b\mathbf{A} + \mathbf{A}\times\mathbf{B} \quad . \quad \text{.........} \quad (2)$$

(b) By Mixed Numbers

Mixed number[5] α is the sum of a scalar x and a vector **A**, i.e. $\alpha = x + \mathbf{A}$. The product of two mixed numbers is defined as[5]

$$\alpha\beta = (x + \mathbf{A})(y + \mathbf{B}) = xy + \mathbf{A}.\mathbf{B} + x\mathbf{B} + y\mathbf{A} + i\mathbf{A}\times\mathbf{B} \quad . \qquad \ldots\ldots\ldots (3)$$

Let $\quad A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} = \mathbf{A}$ and $B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k} = \mathbf{B}$

then $\quad \alpha\beta = (x + \mathbf{A})(y + \mathbf{B}) = (x + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})(y + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k})$

or, $\alpha\beta = xy + x B_1\mathbf{i} + x B_2\mathbf{j} + x B_3\mathbf{k} + yA_1\mathbf{i} + A_1B_1\mathbf{ii} + A_1B_2\mathbf{ij} + A_1B_3\mathbf{ik} + yA_2\mathbf{j} + A_2B_1\mathbf{ji} + A_2B_2\mathbf{jj} + A_2B_3\mathbf{jk} + y A_3\mathbf{k} + A_3B_1\mathbf{ki} + A_3B_2\mathbf{kj} + A_3B_3\mathbf{kk}$

or, $\alpha\beta = (xy + A_1B_1\mathbf{ii} + A_2B_2\mathbf{jj} + A_3B_3\mathbf{kk}) + x (B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) + y(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) + (A_2B_3\mathbf{jk} + A_3B_2\mathbf{kj}) + (A_3B_1\mathbf{ki} + A_1B_3\mathbf{ik}) + (A_1B_2\mathbf{ij} + A_2B_1\mathbf{ji})$

or, $\alpha\beta = (xy + A_1B_1\mathbf{ii} + A_2B_2\mathbf{jj} + A_3B_3\mathbf{kk}) + x \mathbf{B} + y\mathbf{A} + (A_2B_3\mathbf{jk} + A_3B_2\mathbf{kj}) + (A_3B_1\mathbf{ki} + A_1B_3\mathbf{ik}) + (A_1B_2\mathbf{ij} + A_2B_1\mathbf{ji})$. ……............................. (4)

The left side of equation (3) and (4) are same, therefore the right side of these two equations must be same. Therefore we can write

$xy + \mathbf{A}.\mathbf{B} + x\mathbf{B} + y\mathbf{A} + i\mathbf{A}\times\mathbf{B} = (xy + A_1B_1\mathbf{ii} + A_2B_2\mathbf{jj} + A_3B_3\mathbf{kk}) + x \mathbf{B} + y\mathbf{A} + (A_2B_3\mathbf{jk} + A_3B_2\mathbf{kj}) + (A_3B_1\mathbf{ki} + A_1B_3\mathbf{ik}) + (A_1B_2\mathbf{ij} + A_2B_1\mathbf{ji})$

or, $\mathbf{A}.\mathbf{B} + i\mathbf{A}\times\mathbf{B} = (A_1B_1\mathbf{ii} + A_2B_2\mathbf{jj} + A_3B_3\mathbf{kk}) + (A_2B_3\mathbf{jk} + A_3B_2\mathbf{kj}) + (A_3B_1\mathbf{ki} + A_1B_3\mathbf{ik}) + (A_1B_2\mathbf{ij} + A_2B_1\mathbf{ji})$. ………………………. .... (5)

Let $\quad \mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 1$

and $\quad \mathbf{ij} = i\,\mathbf{k},\ \mathbf{jk} = i\,\mathbf{i},\ \mathbf{ki} = i\,\mathbf{j},\ \mathbf{ji} = -\,i\,\mathbf{i},\ \mathbf{kj} = -\,i\,\mathbf{i},\ \mathbf{ik} = -\,i\,\mathbf{j}$

where $i = \sqrt{(-1)}$.

Using these values in equation (5) we see that the left side and right side of this equation are same. Therefore from the mixed number algebra we get the properties of **i**, **j** and **k** are

$$\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 1 \text{ i.e. } \mathbf{i}^2 = 1, \mathbf{j}^2 = 1, \mathbf{k}^2 = 1$$

and $\quad \mathbf{ij} = i\,\mathbf{k},\ \ \mathbf{jk} = i\,\mathbf{i},\ \ \mathbf{ki} = i\,\mathbf{j},\ \ \mathbf{ji} = -\,i\,\mathbf{i},\ \ \mathbf{kj} = -\,i\,\mathbf{i},\ \ \mathbf{ik} = -\,i\,\mathbf{j}$

where $i = \sqrt{(-1)}$.

Again mixed number $\alpha = x + \mathbf{A} = (x + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$ where x, A_1, A_2 and A_3 are scalar values and **i**, **j** and **k** are unique Mixed Numbers with the properties that

$$\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 1 \text{ i.e. } \mathbf{i}^2 = 1, \mathbf{j}^2 = 1, \mathbf{k}^2 = 1$$

and $\mathbf{ij} = i\,\mathbf{k}$, $\mathbf{jk} = i\,\mathbf{i}$, $\mathbf{ki} = i\,\mathbf{j}$, $\mathbf{ji} = -i\,\mathbf{i}$, $\mathbf{kj} = -i\,\mathbf{i}$, $\mathbf{ik} = -i\,\mathbf{j}$

where $i = \sqrt{(-1)}$.

This is also clearly an extension of the complex number system, where the complex numbers are those Mixed Numbers that have $A_2 = A_3 = 0$ and the real numbers are those that have $A_1 = A_2 = A_3 = 0$.

2. Conclusion

Mixed numbers are the extension of complex numbers like quaternions but the properties of unique quaternions and the properties of unique mixed numbers are different.

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Journal Home Page