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Invariance of the Counted Number of Photons: Some Didactic Opportunities

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Abstract: *It has long been assumed that time dilation is necessary in the derivation of the transformation equation for the energy of the photon. In this paper we demonstrate in a single scenario how we can apply length contraction to the volume in which energy carried by the electromagnetic wave is confined in order to save the invariance of the number of photons.*
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1. Plane electromagnetic wave

Two published papers [1], [2] have in common the same scenario but different purposes: the first derives a transformation equation for the energy carried by a single photon, whereas the second answers the question: "is the number of photons a classical invariant?" The relativistic invariance of the universal constants and of the counted number of stable particles is one of the important consequences of the relativistic postulate: *The true laws of physics are the same in all inertial reference frames* [3].

It was shown that length contraction applied to the volume in which energy is carried by an electromagnetic wave is confined leads to the conclusion that the number of photons is not an invariant [2] and that time dilation, applied in a questionable manner, leads to the correct transformation equation for the energy carried by a photon. The purpose of our paper is to show, from a single scenario, how we can apply length contraction to the volume in which energy carried by the electromagnetic wave is confined in order to save the invariance of the number of photons and that *time dilation is not compulsory in the derivation of the transformation equation for the energy of the photon.*

Consider a plan monochromatic wave, propagating in the positive direction of the $O'X'$ axis of the $S'(X'O'Y')$ inertial reference frame as it is shown in Fig.1. A shutter 1, perpendicular to the direction of propagation is removed for a *proper* time interval τ', measured as a difference between the readings of a clock C_0' at rest in S' and located where the shutter is removed. Let W' and N' be the energy carried by the electromagnetic wave and the carried number of photons respectively, through the surface A' opened by the shutter for a time interval τ'. Energy W' and the N' photons are located inside a volume:

$$V_c' = A'c\tau' \quad . \tag{1}$$

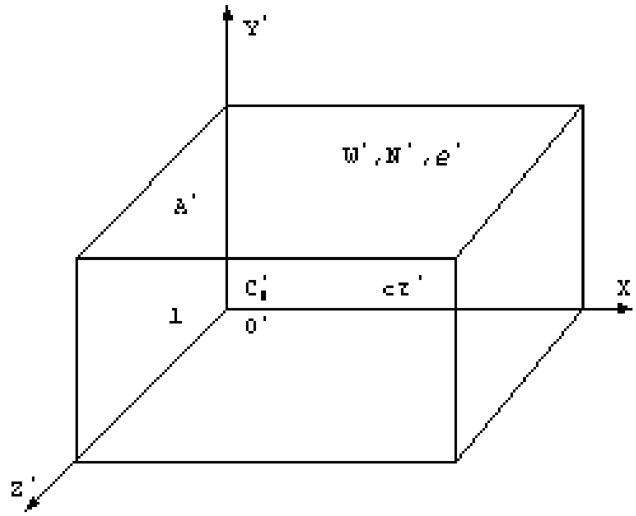

Figure 1. The volume $V_c^{'} = A^{'}c\tau^{'}$ in which the energy carried by the plane electromagnetic wave is confined.

The energy density in that volume is given by:

$$\rho^{'} = \frac{W^{'}}{V_c^{'}} = \frac{N^{'}Q^{'}}{V_c^{'}} = \frac{N^{'}Q^{'}}{A^{'}c\tau^{'}} \quad , \tag{2}$$

with $Q^{'}$ representing the energy carried by a single photon $(W^{'} = N^{'}Q^{'})$. The volume $V_c^{'}$ moves with velocity c and so no photon enters or leaves that volume.

We consider now the same experiment from an inertial reference frame $S(XOY)$, relative to which reference frame $S^{'}(X^{'}O^{'}Y^{'})$ moves with velocity $v = \beta c$ in the positive direction of the common $OX(O^{'}X^{'})$ axes. In accordance with the relativistic postulate, Equation (2) reads in that frame

$$\rho = \frac{W}{V_c} = \frac{NQ}{V_c} = \frac{NQ}{Ac\tau} \quad , \tag{3}$$

the unprimed quantities in Equation (3) having the same physical meaning as the primed in Equation (2).

It is well known [1] that in the plane electromagnetic wave the following transformation equations hold:

$$W = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} W^{'} = kW^{'} \quad , \tag{4}$$

$$\rho = k^2\rho^{'} . \tag{5}$$

The shutter being perpendicular to the direction of relative motion its surface is a relativistic invariant $(A = A^{'})$. Combining Equations (2) and (3) we obtain first

$$\frac{V_c}{V_c^{'}} = \left(\frac{W}{W^{'}}\right)\left(\frac{\rho^{'}}{\rho}\right) \quad , \tag{6}$$

leading to the fact that the volumes V_c and $V_c^{'}$ should transform as:

$$V_c = k^{-1}V_c^{'} . \tag{7}$$

In order to justify Equation (7) consider a rod of proper length L_0 moving with velocity u relative to $S(XOY)$ but with velocity u' relative to $S'(X'O'Y')$. The velocities u and u' are related by the addition law of relativistic velocities:

$$u = \frac{u' + v}{1 + u'vc^{-2}} \quad . \tag{8}$$

The length of the rod measured in S is

$$L = L_0\left(1 - u^2c^{-2}\right)^{1/2} \quad , \tag{9}$$

whereas measured from S' it is

$$L' = L_0\left(1 - u'^2c^{-2}\right)^{1/2} \quad . \tag{10}$$

Equations (8), (9), and (10) lead to:

$$L = L'\frac{1 - v^2c^{-2}}{1 + u'vc^{-2}} \quad . \tag{11}$$

If the rods move with velocity c as the sides of volumes V_c and V_c' move, Equation (11) becomes

$$L_c = k^{-1}L_c' \quad . \tag{12}$$

Multiplying both sides of Equation (12) with $Ac = A'c$ we recover Equation (7), which at its turn leads to:

$$\frac{V_c}{V_c'} = \frac{Ac\tau}{A'c\tau'} \quad . \tag{13}$$

leading to:

$$\tau = k^{-1}\tau' \quad . \tag{14}$$

Combining Equations (2) and (3) we also have

$$\frac{N}{N'} = \left(\frac{V}{V'}\right)\left(\frac{Q'}{Q}\right)\left(\frac{\rho}{\rho'}\right) \tag{15}$$

and

$$\frac{N}{N'} = \left(\frac{\tau}{\tau'}\right)\left(\frac{Q'}{Q}\right)\left(\frac{\rho}{\rho'}\right) \quad . \tag{16}$$

Replacing Equation (13) in Equation (15) we obtain the expected result $N = N'$, avoiding the paradox invoked in [2].

Margaritondo [1] considers that the time intervals τ and τ' are related by the time dilation formula $\left(\tau = \left(1 - \beta^2\right)^{-1/2}\tau'\right)$ which would lead to the paradox according to which N and N' are not equal to each other. As we see Equation (14) avoids that paradox. In order to justify Equation (14) we can invoke the following arguments:

- τ and τ' represent proper time intervals which are related by the Doppler factor k as prescribed by Equation (14),
- τ and τ' can represent the periods of the electromagnetic oscillations taking place in the wave $\left(\tau = T, \tau' = T'\right)$ which are related by $T = k^{-1}T'$ (17),
- - $cT = \lambda$ and $cT' = \lambda'$ represent wavelengths which are related by $\lambda = k^{-1}\lambda'$ (18),
- - $N/\tau = \nu_c$ and $N'/\tau' = \nu_c'$ represent counting frequencies of the emitted photons, resulting that $\nu_c/\nu_c' = k$ and $\tau/\tau' = k^{-1}$ [4].

2. Spherical electromagnetic wave

With the knowledge acquired above, we revisit the way in which transformation equations for physical quantities introduced in order to characterize the radiation emitted by a point-like source of electromagnetic radiation. Such a source $L^{'}$ located at the origin $O^{'}$ of its rest frame $S^{'}(X^{'}O^{'}Y^{'})$ emits during a proper time interval $d\tau^{'}$ an energy of $dW^{'}$ and $dN^{'}$ photons with the energy $Q^{'}$ for each of them. The energy is emitted inside a solid angle $d\Omega^{'}$ centered around a direction which makes an angle $\theta^{'}$ with the positive direction of the $OX(O^{'}X^{'})$ axes. After a time $t^{'}$ of propagation the emitted energy is stored inside the volume (Fig.2)

$$dV^{'} = \left(ct^{'}\right)^{2} d\Omega^{'} c d\tau^{'} \tag{19}$$

having an energy density

$$\rho^{'} = \frac{Q^{'} d^{2} N^{'}}{\left(ct^{'}\right)^{2} d\Omega^{'} c d\tau^{'}} . \tag{20}$$

When considered from the $S(XOY)$ frame, the density of energy is

$$\rho = \frac{Q d^{2} N}{(ct)^{2} d\Omega c d\tau} \tag{21}$$

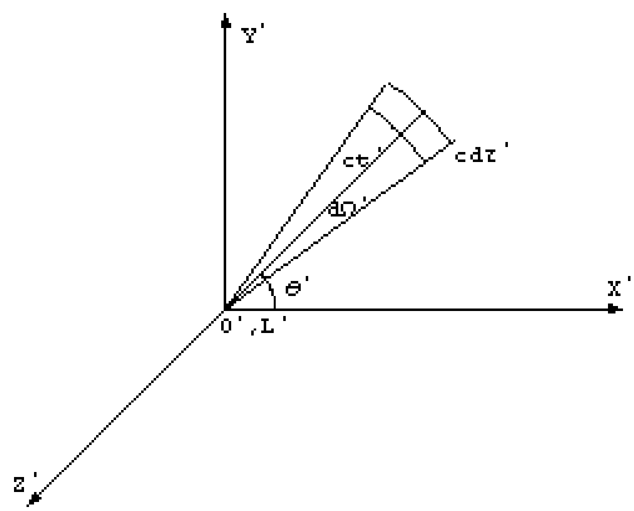

Figure 2. The volume $\left(ct^{'}\right)^{2} d\Omega^{'} c d\tau^{'}$ in which the energy emitted by the point-like source $L^{'}$ during a time interval $d\tau^{'}$ is confined.

There are sufficient reasons to consider that the physical quantities in the right sides of Equations (20) and (21) transforms as [4]:

$$\rho = D^{2} \rho^{'} \quad , \tag{22}$$

$$Q = D Q^{'} \quad , \tag{23}$$

$$d\Omega = D^{-2} d\Omega^{'} \quad , \tag{24}$$

$$t = D t^{'} \quad , \tag{25}$$

where D represents the non-longitudinal Doppler factor, given by

$$D = \frac{1+\beta\cos\theta'}{(1-\beta^2)^{1/2}} \qquad . \tag{26}$$

The result is that the number of emitted photons should transform as

$$\frac{d^2N}{d^2N'} = \left(\frac{\rho}{\rho'}\right)\left(\frac{ct}{ct'}\right)\left(\frac{d\Omega}{d\Omega'}\right)\left(\frac{Q'}{Q}\right)\left(\frac{d\tau}{d\tau'}\right) = D\left(\frac{d\tau}{d\tau'}\right). \qquad . \tag{27}$$

The invariance of the counted number of photons requires that emission times transform as

$$\frac{d\tau}{d\tau'} = D^{-1} \tag{28}$$

and not as

$$\frac{d\tau}{d\tau'} = \frac{1}{(1-\beta^2)^{1/2}} \qquad , \tag{29}$$

if time dilation is taken into account.

The important result is that physical quantities introduced in order to characterize the emission and the reception of radiant energy and defined as a combination of energy, emission and reception times, propagation time, volume, surface and solid angle transform via transform factor D^n [1], [4], [5] , (n integer) and not as $D^{n-1}\gamma$ as proposed in [6], [7].

We underline that even using time dilation during the derivations [1], [5], the final result is that transformation is performed via the factor D^n, due to the fact that the factor γ is compensated by the way in which other physical quantities are considered to transform.

3. Conclusions

Physicists warn us that in many cases, time dilation and length contraction, not properly handled, lead to embarrassing paradoxes. Our paper is an illustration of that fact, showing how to avoid them.

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