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Natura non facit saltus: An Uncommon Way to Relativistic Dynamics

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Abstract: *The relativistic transformation equations for mass, momentum and energy are derived without using conservation laws but based on the idea that nature ensures a smooth transition from the physical properties of a tardyon to those of a photon.*
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1. Introduction

The student who has studied classical physics knows Galileo's postulate: The laws of physics are the same inside a laboratory moving at a constant velocity as they are in a laboratory at rest (*principle of indistinguishability of inertial frame of reference*). Experiments performed by Bertozzi[1] convinced him there is an upper limit, c, of velocity up to which particles like electrons can be accelerated ($u<c$). Such particles are called *tardyons*. Classical mechanics taught him to characterize the properties of a tardyon by mass m, by its momentum

$$p = mu \tag{1}$$

and by its kinetic energy

$$E_k = \frac{1}{2}mu^2 \tag{2}$$

E_k is the unique energy which a free tardyon can possess. Conservation of momentum is the basic postulate commonly used in deducing dynamical results in special relativity[2]. Classical electromagnetism teaches us that an electromagnetic wave has an energy E_e and a momentum p_e related by

$$p_e = \frac{E_e}{c} \quad . \tag{3}$$

Considering that electromagnetic energy and momentum are carried by N photons[3], particles which move with the same velocity c relative to all inertial reference frames, c is equal to the velocity with which the associated electromagnetic wave propagates in empty space. If p_f and E_f represent momentum and energy of photon respectively, then $p_e = Np_f$ and $E_e = NE_f$, from Equation (3) leading to the following relationship between them

$$p_f = \frac{E_f}{c} \ . \tag{4}$$

The concept of mass is also extended to photon m_f imposing the relationship[4]

$$m_f = \frac{p_f}{c} \tag{5}$$

$$m_f = \frac{E_f}{c^2} \ . \tag{6}$$

Relativistic kinematics teaches the learner that many physical quantities like length and time have different properties than in classical kinematics. Relativistic dynamics operates with the concept of time and so it is natural to ask if classical concepts like mass, momentum and energy are not affected.

The subject of physical quantities introduced in order to characterize the dynamic properties of a tardyon (rest mass, relativistic mass, relativistic momentum, rest energy, relativistic energy and relativistic kinetic energy) has long been a topic for debate. It was generated by the different ways in which the fundamental equations are derived, using for each of them different scenarios[5,6,7,8]. Collisions and conservation laws of momentum and energy play a fundamental part in teaching relativistic dynamics. Many derivations are based on collisions between tardyons and photons[9].

In the approach we propose, all equations of relativistic dynamics are derived from a single scenario, consequently applying the relativistic postulate and the fact that all formulas used in relativistic dynamics should lead for ***u***=***c*** to formulas which describe the dynamic properties of the photon, i.e.

$$\lim_{u \to c} p = p_f \ . \tag{7}$$

If ***E*** represents the energy of a tardyon, Equation (7) is satisfied if equation

$$p = \frac{Eu}{c^2} \tag{8}$$

can be considered as an extension of Leibniz's idea "Natura non facit saltus," nature ensuring a smooth transition from the properties of the tardyon to the properties of the photon.

2. Relativistic dynamics of a tardyon in one-space dimensions

The scenario we propose involves a tardyon moving with velocity $\vec{u}(u_x, u_y = 0)$ relative to an inertial reference frame *S(XOY)* in the positive direction of the *OX* axis. Let *S'(X'O'Y')* be a second inertial reference frame moving with velocity $\vec{v}$ in the positive direction of the common *OX(O'X')* axes. The axes of the two frames are parallel to each other. At an instant of time *t=t'=0*, the origins *O* and *O'* are located at the same point in space. Let $\vec{u}'(u'_x, u'_y = 0)$ be the velocity of the same tardyon relative to the *S'* frame. In accordance with Equation (8) and with the relativistic postulate, the momentum of the tardyon detected from *S* and *S'* is

$$p_x = \frac{Eu_x}{c^2} \tag{9}$$

$$p'_x = \frac{E'u'_x}{c^2} \tag{10}$$

respectively, primed quantities being measured from the *S'* frame. Combining equations (9) and (10) we obtain

$$\frac{p_x}{E} = \frac{p'_x}{E'}\frac{u_x}{u'_x} = \frac{p'_x}{E'}\frac{1 + vu'^{-1}_x}{1 + u'_x c^{-1}} \tag{11}$$

$$\frac{p_x^{'}}{E^{'}} = \frac{p_x}{E}\frac{u_x^{'}}{u_x} = \frac{p_x}{E}\frac{1 - vu_x^{-1}}{1 + u_x c^{-1}} \tag{12}$$

where we have expressed the right hand side of Equation (11) as a function of physical quantities measured in *S'* and the right hand side of Equation (12) as a function of physical quantities measured in *S* via the addition law of relativistic velocities:

$$u_x = \frac{u_x^{'} + v}{1 + vu_x^{'}c^{-2}} \tag{13}$$

$$u_x^{'} = \frac{u_x - v}{1 + vu_x c^{-2}} \quad . \tag{14}$$

Equation (11) leads to

$$p_x = f'(v) p_x^{'} (1 + vu_x^{'-1}) \tag{15}$$

$$E = f'(v) E'(1 + vu_x^{'} c^{-2}) \quad , \tag{16}$$

and that

$$p_x^{'} = f(v) p_x (1 - vu_x^{-1}) \tag{17}$$

$$E' = f(v) E(1 + vu_x c^{-2}) \quad , \tag{18}$$

as suggested by Equation (12), *f(v)* and *f'(v)* representing unknown functions of the relative velocity ***v*** but not of the physical quantities involved in the transformation process, due to the linearity of the transformation equations. Combining Equations (16) and (18) we obtain

$$f(v) f'(v) = (1 - v^2 c^{-2})^{-1} = \gamma^{-2} . \tag{19}$$

Consider first the case when the tardyon is at rest in *S'* moving with velocity ***v*** relative to *S* ($u_x = c, u_x^{'} = 0, u_y^{'} = u_y = 0$). Under such conditions Equation (16) leads to

$$E_{\substack{u_x^{'}=0 \\ u_x=0}} = f'(v) E'_{u_x^{'}=0} \quad . \tag{20}$$

If the same particle is at rest in the *S* frame, it moves with velocity -***v*** relative to *S'* (as required by symmetry). Now $u_x = 0, u_y = 0$, Equation (18) telling us that:

$$E'_{\substack{u_x=0 \\ u_x^{'}=-v}} = f(v) E_{u_x=0} \quad . \tag{21}$$

Consistency requires that

$$E_{u_x=0} = E^{'}_{u_x^{'}=0} \quad , \tag{22}$$

and

$$E_{\substack{u_x^{'}=0 \\ u_x=v}} = E'_{\substack{u_x=0 \\ u_x^{'}=-v}} \quad , \tag{23}$$

leading to

$$f(v) = f'(v) = \gamma = (1 - u^2 c^{-2})^{1/2} \quad . \tag{24}$$

The first important result is that a tardyon at rest in an inertial reference frame should be measured from a frame having a rest energy E_0 given by

$$E_0 = E_{u_x=0} = E'_{u_x^{'}=0} \quad , \tag{25}$$

which has no counterpart in classical mechanics.

Being at rest in S', the energy of the tardyon measured from S is given by ($u_x^{'} = 0, u_x = v$)

$$E\Big|_{\substack{u_x'=0\\u_x=v}} = E'\big|_{u_x'=0} = E_0 \quad . \tag{26}$$

In general when a tardyon moves with velocity ***u*** relative to an inertial reference frame S then its energy in that frame is

$$E = \frac{E_0}{(1-u^2c^{-2})^{1/2}} \quad , \tag{27}$$

or in general, if a particle moves with velocity $\vec{u}$ relative to the *S* frame its momentum measured from that frame is

$$\vec{p} = E_0 c^{-2}\gamma_u \vec{u} \quad . \tag{28}$$

Equation (27) confirms the fact that for ***u*=*0*, *E*=*E*$_0$**.

In general case, when the tardyon moves with velocity $\boldsymbol{u_x}$ relative to S and with velocity $\boldsymbol{u_x'}$ relative to S', Eqs. (15) and (16) become

$$p_x = \gamma p_x'(1+vu_x'^{-1}) = \gamma(p_x' + vc^{-2}E') \quad , \tag{29}$$

and

$$E = \gamma E'(1+vu_x'c^{-2}) = \gamma(E' + vc^{-2}p_x') \quad . \tag{30}$$

The tardyon moving with velocity ***u*** relative to *S* has when measured from that frame a kinetic energy

$$E_k = E - E_0 = E_0(\gamma_u - 1) \tag{31}$$

as the single energy a free tardyon can possess.

Starting with the obvious identity $\gamma_u^2\gamma_u^{-2} = 1$ and multiplying both its sides with E_0^2 we obtain taking into account the notations introduced above

$$E^2 - p_x^2c^2 = E_0^2 \quad . \tag{32}$$

3. Relativistic dynamics in two-space dimensions

The tardyon moves now along a direction which makes an angle θ with the positive direction of the *OX* axis, starting from the origin *O* of the *S(XOY)* reference frame. Its velocity is $\vec{u}(u_x, u_y)$. When detected from *S'(X'O'Y')* it moves with velocity $\vec{u}'(u_x', u_y')$ along a direction which makes an angle θ' with the positive direction of the *O'X'* axis. The tardyon starts at an instant of time *t*=*t'*=*0* from the point where the origins *O* and *O'* are instantly located. Relativistic kinematics teaches us that the *OY(O'Y'*) components of the velocity are related by

$$u_y = \frac{\gamma^{-1}u_y'}{1+u_x'vc^{-2}} \quad . \tag{33}$$

The magnitudes of the velocity are related by

$$u = (u_x^2 + u_y^2)^{1/2} = \frac{u'(\gamma^{-2}\sin^2\theta' + (1+vu'^{-1})^2)^{1/2}}{1+u'vc^{-2}\cos\theta'} \quad . \tag{34}$$

The angles θ and θ' are related by

$$tg\theta = \frac{u_y}{u_x} = \frac{\gamma^{-1}\sin\theta'}{\cos\theta' + vu'^{-1}} \quad . \tag{35}$$

The *OY (O'Y')* components of the momentum transform as

$$p_y = \frac{E_0 u_y}{c^2(1-u^2c^{-2})^{1/2}} = p_y^{'} \quad , \tag{36}$$

leading to this component which is a relativistic invariant.

The energy transforms in the case of two-space dimensions problem in accordance with Equation (30) being independent of the *OY(O'Y')* components of velocity and momentum. The *OX(O'X')* component of the momentum transforms in accordance with Equation (29).

4. "Natura non facit saltus" at work: Dynamics of the photon

As we have mentioned in introduction, the equations which describe the dynamic properties of the tardyon should describe for $u \to c$ and $u^{'} \to c$ the dynamic properties of the photon, the particle which moves with the same velocity relative to all inertial frames. We replace now the tardyon with a photon, considering the same initial conditions as in Section 3. The photon moves along the directions θ_f and $\theta_f^{'}$ when detected from *S* and *S'* respectively. In accordance with Equation (35) the two angles are related by

$$tg\theta_f = \frac{u_y}{u_x} = \frac{\gamma^{-1}\sin\theta_f^{'}}{\cos\theta_f^{'} + vc^{-1}} \quad . \tag{37}$$

The transformation equation for the energy of the photon is obtained from Equation (30) making there $u_x^{'} = c\cos\theta_f^{'}$

$$E_f = E_f^{'} \frac{1 + vc^{-1}\cos\theta_f^{'}}{(1-u^2c^{-2})^{1/2}} \quad , \tag{38}$$

where the transformation equations for the components of its momentum are obtained from Equations (29) and (36)

$$p_{f,x} = p_{f,x}^{'} \frac{1 + vc^{-1}}{(1-u^2c^{-2})^{1/2}} \tag{39}$$

$$p_{f,x} = p_{f,x}^{'} \quad , \tag{40}$$

with the magnitude of the momentum transforming as

$$p_f = p_f^{'} \frac{1 + vc^{-1}\cos\theta_f^{'}}{(1-u^2c^{-2})^{1/2}} \quad . \tag{41}$$

The fundamental equation of relativistic dynamics (Equation (32)) tells us that

$$E_{0,f}^2 = E_f^2 - p_f^2c^2 = 0 \quad . \tag{42}$$

From Equation (42) it follows that the rest energy of the photon is zero.

5. Back to the concept of mass

We have avoided so far in our derivations the concept of mass. Coming from classical mechanics we would start by expressing the momentum of the tardyon as

$$p = mu \quad , \tag{43}$$

being obliged to consider, in order to be in accordance with the results obtained above, that *m* and *E* should be related by

$$m = Ec^{-2} \tag{44}$$

in the *S* frame, and by

$$m^{'} = E^{'}c^{-2} \tag{45}$$

in the S' frame. The masses m and m' are related by Equation (30)

$$m = m' \frac{1 + vu'_x c^{-2}}{(1 - u^2 c^{-2})^{1/2}} = \frac{m' + vp'_x c^{-2}}{(1 - u^2 c^{-2})^{1/2}} \tag{46}$$

The concept of rest energy has its correspondent in the concept of rest mass m_0 (mass of the tardyon measured by an observer relative to whom it is in state of rest), related by

$$m_0 = E_0 c^{-2}. \tag{47}$$

In the above equation m_0 represents the Newtonian mass as well. Equation (44) is known as Einstein's mass-energy relationship and we can express it as

$$E = mc^2 = \frac{m_0 c^2}{(1 - u^2 c^{-2})^{1/2}} = \frac{E_0}{(1 - u^2 c^{-2})^{1/2}} \quad . \tag{48}$$

Knowing that mass denotes inertia, a reluctance to undergo a change in velocity, we see that energy can serve the same purpose, being a measure of inertia of a body.

6. Conclusions

Our approach to relativistic dynamics offers to the physicist the following list of alternatives: rest mass, relativistic mass, momentum, rest energy and energy, each with a well-defined physical meaning. Some physicists, especially those who enjoy "Gedanken experiments" prefer to use all of them. Others may avoid the use of the concept of relativistic mass whereas others avoid any concept of mass altogether.

Knowing that the equations derived above can be obtained from various scenarios which involve collisions between tardyons or between tardyons and photons and use conservation of momentum and energy (mass), we can be sure that using the equations derived above we obtain results in accordance with them. For those who avoid the concept of mass it is the conservation of momentum and energy that works, whereas for those who accept the concept of mass it is the conservation of momentum that leads to correct results. In the first case we say that energy transforms in terms of energy and in the second case we say that mass transforms in mass but we never say that mass transforms in energy and reverse.

Our approach informs the learner about the conventions used in teaching relativistic dynamics, being time saving and convincing him that physics is not a cluster of disconnected units of knowledge, in accordance with the aims claimed by physics educators.

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